DE Exam (with solutions)
Texas A&M High School Math Contest
November 12, 2022

1. It is known that $(\sqrt{2}-1)^4 = \sqrt{N} - \sqrt{N-1}$, where N is an integer. Find N.

Answer: 289.

Using the formula $(a-b)^2 = a^2 - 2ab + b^2$, we obtain $(\sqrt{2}-1)^2 = (\sqrt{2})^2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2}$. Then $(\sqrt{2}-1)^4 = (3-2\sqrt{2})^2 = 3^2 - 2\cdot 3\cdot 2\sqrt{2} + (2\sqrt{2})^2 = 17 - 12\sqrt{2}$. Alternatively, we can use the binomial formula:

$$(\sqrt{2}-1)^4 = (\sqrt{2})^4 - 4(\sqrt{2})^3 + 6(\sqrt{2})^2 - 4\sqrt{2} + 1 = 4 - 8\sqrt{2} + 12 - 4\sqrt{2} + 1 = 17 - 12\sqrt{2}.$$

Now we observe that $17 - 12\sqrt{2} = \sqrt{17^2} - \sqrt{12^2\cdot 2} = \sqrt{289} - \sqrt{288}$. Thus $N = 289$.

2. A parallelogram has sides of length 2 and 3. One of its diagonals has length 4. Find the length of the other diagonal.

Answer: $\sqrt{10}$.

Let B and D be endpoints of the diagonal of length 4. Let A and C be the other two vertices of the parallelogram denoted so that $|AB| = |CD| = 2$ and $|AD| = |BC| = 3$. Applying the Law of Cosines to the triangle ABD, we obtain $|BD|^2 = |AB|^2 + |AD|^2 - 2|AB|\cdot|AD|\cos\angle BAD$. Then

$$\cos\angle BAD = \frac{|AB|^2 + |AD|^2 - |BD|^2}{2|AB|\cdot|AD|} = \frac{2^2+3^2-4^2}{2\cdot 2\cdot 3} = -\frac{1}{4}.$$

The angles BAD and ABC are adjacent angles of a parallelogram. Therefore $\angle BAD + \angle ABC = \pi$, which implies that $\cos\angle ABC = -\cos\angle BAD = 1/4$. Applying the Law of Cosines to the triangle ABC, we obtain

$$|AC|^2 = |AB|^2 + |BC|^2 - 2|AB|\cdot|BC|\cos\angle ABC = 2^2 + 3^2 - 2\cdot 2\cdot 3\cdot\frac{1}{4} = 10.$$

Thus the diagonal AC has length $\sqrt{10}$.

3. Find the probability that an integer number chosen randomly in the range from 1 to 10^5 (inclusive) has exactly three odd digits (not necessarily distinct).

Answer: $\frac{5}{16} = 0.3125$ or alternatively 31.25%.

Let S be the set of all integers between 1 and 10^5 that have exactly three odd digits. Then the probability equals $N/10^5$, where N is the number of elements in S. Note that 100000 does not have three odd digits. Any integer between 0 and 99999 can be written (uniquely) as a 5-digit number $\overline{d_1d_2d_3d_4d_5}$, where one or more leading digits can be 0. Since 0 is not odd, the number of odd digits in this notation is the same as in the standard decimal notation. To construct an element of the set S in the nonstandard notation, we first specify which digits are going to be odd and which are going to be even. We choose three positions (out of five) for odd digits and leave the other two for even digits. The number of possible choices is

$$\binom{5}{3} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = 10.$$

Once positions of odd and even digits are specified, there are 5 choices for each position (1, 3, 5, 7 or 9 for odd digits; 0, 2, 4, 6 or 8 for even digits). Since all choices are independent, this produces 5^5 different integers. Overall, there are $10 \cdot 5^5$ numbers in the set S. Hence the probability equals $10 \cdot 5^5/10^5 = 10/2^5 = 5/16$.

4. Evaluate the following expression: $\cos\Big(\arcsin\frac{4}{5}\Big) + \arccos\Big(\sin\frac{4}{5}\Big)$.

Answer: $\frac{\pi}{2} - \frac{1}{5}$.

Let $\alpha = \arcsin(4/5)$. Then $|\alpha| \le \pi/2$ and $\sin\alpha = 4/5$. We obtain $\cos^2\alpha = 1 - \sin^2\alpha = 1 - (4/5)^2 = 9/25$. Note that $\cos\alpha \ge 0$ since $|\alpha| \le \pi/2$. Hence $\cos\alpha = \sqrt{9/25} = 3/5$.

Note that $0 < 4/5 < 1 < \pi/2$. It follows that $0 < \sin(4/5) < 1 < \pi/2$. Therefore the angle $\beta = \arccos(\sin(4/5))$ lies between 0 and $\pi/2$. We have $\cos\beta = \sin(4/5) = \cos(\pi/2 - 4/5)$. Since $4/5$ lies between 0 and $\pi/2$, so does $\pi/2 - 4/5$. Hence the equality of cosines of the angles β and $\pi/2 - 4/5$ implies that the angles are the same: $\beta = \pi/2 - 4/5$. Thus

$$\cos\Big(\arcsin\frac{4}{5}\Big) + \arccos\Big(\sin\frac{4}{5}\Big) = \frac{3}{5} + \frac{\pi}{2} - \frac{4}{5} = \frac{\pi}{2} - \frac{1}{5}.$$

5. Find the largest integer x satisfying the inequality $x < \sqrt{|x - 2022|}$.

Answer: 44.

The inequality clearly holds for all negative numbers. For any $x \ge 0$, it is equivalent to $x^2 < |x - 2022|$. We are going to consider two cases.

Case 1: $0 \le x < 2022$. In this case $|x - 2022| = -(x - 2022) = 2202 - x$ and the inequality becomes

$$x^2 < 2022 - x \iff x^2 + x - 2022 < 0 \iff \Big(x + \frac{1}{2}\Big)^2 < 2022 + \frac{1}{4}$$
$$\iff -\frac{1}{2} - \sqrt{2022 + \frac{1}{4}} < x < -\frac{1}{2} + \sqrt{2022 + \frac{1}{4}}.$$

It follows that in this case the largest integer satisfying the inequality is the largest integer that is less than $-1/2 + \sqrt{2022 + 1/4}$. We obtain that $44^2 = 1936$ and $45^2 = 2025$. Furthermore, $44.5^2 = 1980.25$. Therefore $44 < -1/2 + \sqrt{2022 + 1/4} < 45$ so that the largest integer is 44.

Case 2: $x \ge 2022$. In this case $|x - 2022| = x - 2022$ and the inequality becomes

$$x^2 < x - 2022 \iff x^2 - x + 2022 < 0 \iff \Big(x - \frac{1}{2}\Big)^2 < \frac{1}{4} - 2022,$$

which has no solutions.

6. Suppose T is a triangle in the plane with sides of length 2, 3 and 4. Let F be the figure that consists of all points of T as well as all points at distance at most 1 from the triangle. Find the perimeter of the figure F.

Answer: $2\pi + 9$.

The figure F can be cut into 7 pieces: the triangle T, three rectangles with one side of length 1, and three circular sectors of circles of radius 1. Each rectangle shares a side with the triangle T

and each sector is centered at a vertex of T. The boundary of F consists of three line segments, each of which is a translation of one side of T, and three circular arcs, which are curvilinear parts of boundaries of the sectors. It follows that the perimeter of F equals $p + s$, where p is the perimeter of the triangle T (equal to 9) and s is the sum of central angles of the sectors.

At each vertex of the triangle T, four pieces of the figure F meet: the triangle itself, two rectangles and one sector. Since the full angle at the vertex equals 2π and every angle of rectangles equals $\pi/2$, it follows that the central angle of the sector equals $\pi - \alpha$, where α is the angle of T at the same vertex. Let α, β and γ be angles of the triangle T. Then the central angles of the sectors add up to $(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 3\pi - (\alpha + \beta + \gamma)$, which is equal to 2π as the sum of angles of any triangle equals π.

7. A function $f : (0, \infty) \to (0, \infty)$ satisfies a functional equation $x + f(x) = 2f(1/x)$ for all $x > 0$. Find $f(3)$.

Answer: $\dfrac{11}{9}$.

Take any $y > 0$. Using the functional equation first with $x = y$ and then with $x = 1/y$, we obtain $y + f(y) = 2f(1/y)$ and $1/y + f(1/y) = 2f(y)$. From the second equation, $2f(1/y) = 4f(y) - 2/y$. Substituting this into the first equation, we get $y + f(y) = 4f(y) - 2/y$, which implies that $f(y) = (y + 2y^{-1})/3$. In particular, $f(3) = (3 + 2/3)/3 = 11/9$.

8. Find the shortest distance between two circles in the coordinate plane given by equations $x^2 + y^2 = 81$ and $x^2 + y^2 + 6x - 8y + 21 = 0$.

Answer: 2.

The circle with center (x_0, y_0) and radius r is given by the equation $(x - x_0)^2 + (y - y_0)^2 = r^2$. Since $81 = 9^2$, the first circle is centered at the origin O and has radius 9. Since $x^2 + 6x = (x+3)^2 - 9$ and $y^2 - 8y = (y - 4)^2 - 16$, the second equation is equivalent to $(x + 3)^2 + (y - 4)^2 = 9 + 16 - 21$ or $(x + 3)^2 + (y - 4)^2 = 2^2$. Hence the second circle is centered at the point A with coordinates $(-3, 4)$ and has radius 2.

Note that $|OA| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$. Then for any point X on the second circle we have (by the triangle inequality) $|OX| \le |OA| + |AX| = 5 + 2 = 7 < 9$. It follows that the second circle lies completely inside the first circle. Therefore the distance from the point X to the first circle equals $9 - |OX|$. In particular, the point on the second circle closest to the first circle is the one farthest from the origin. This point lies on the intersection of the second circle with the ray OA beyond A. It satisfies $|OX| = |OA| + |AX| = 7$ so that $9 - |OX| = 2$.

9. How many real solutions does the equation $(x^2 - x - 1)^{5x^2 - 19x + 16} = 1$ have?

Answer: 6. $\left[\text{Solutions are } -1,\ 0,\ 1,\ 2,\ \dfrac{19 + \sqrt{41}}{10}, \text{ and } \dfrac{19 - \sqrt{41}}{10}\right]$

If y and z are real numbers, then the equality $y^z = 1$ holds in three cases. Case 1: $y = 1$, z is arbitrary. Case 2: $y = -1$, z is an even positive integer. Case 3: $z = 0$, $y \neq 0$.

Let $y(x) = x^2 - x - 1$ and $z(x) = 5x^2 - 19x + 16$. First we have

$$y(x) = 1 \iff x^2 - x - 2 = 0 \iff (x + 1)(x - 2) = 0 \iff x = -1 \text{ or } x = 2.$$

This gives us two solutions of the given equation: -1 and 2. Next,

$$y(x) = -1 \iff x^2 - x = 0 \iff x(x-1) = 0 \iff x = 0 \text{ or } x = 1.$$

Since $z(0) = 16$ and $z(1) = 2$ are even positive integers, both 0 and 1 are solutions of the given equation. Finally, consider the equation $z(x) = 0$, that is, $5x^2 - 19x + 16 = 0$. This quadratic equation has two roots,

$$x_1 = \frac{19 + \sqrt{41}}{10} \quad \text{and} \quad x_2 = \frac{19 - \sqrt{41}}{10}.$$

Since those two are different from the roots of $x^2 - x - 1$ (which are $\frac{1 \pm \sqrt{5}}{2}$), $y(x_1)$ and $y(x_2)$ are nonzero, so it gives two more solutions.

10. Let P be a pentagon in the coordinate plane with vertices at points $(0,0)$, $(4,0)$, $(5,2)$, $(3,4)$ and $(-1,2)$. Find the area of P.

Answer: 16.

Let $A = (0,0)$, $B = (4,0)$, $C = (5,2)$, $D = (3,4)$ and $E = (-1,2)$. First let us put the pentagon $ABCDE$ into a rectangular box with two vertical and two horizontal sides. The box is bounded by the lines $x = -1$, $x = 5$, $y = 0$ and $y = 4$. The vertices are $F_1 = (-1,0)$, $F_2 = (5,0)$, $F_3 = (5,4)$ and $F_4 = (-1,4)$. To obtain the pentagon from the rectangle, we need to cut off four triangular corners EF_1A, BF_2C, CF_3D and DF_4E. Each of the corners is a right triangle with one horizontal and one vertical leg, which allows to compute their areas easily:

$$\begin{aligned}
\text{area}(EF_1A) &= \frac{1}{2}|F_1A| \cdot |F_1E| = \frac{1}{2} \cdot 1 \cdot 2 = 1, \\
\text{area}(BF_2C) &= \frac{1}{2}|BF_2| \cdot |F_2C| = \frac{1}{2} \cdot 1 \cdot 2 = 1, \\
\text{area}(CF_3D) &= \frac{1}{2}|DF_3| \cdot |CF_3| = \frac{1}{2} \cdot 2 \cdot 2 = 2, \\
\text{area}(DF_4E) &= \frac{1}{2}|F_4D| \cdot |EF_4| = \frac{1}{2} \cdot 4 \cdot 2 = 4.
\end{aligned}$$

The area of the rectangle $F_1F_2F_3F_4$ equals $|F_1F_2| \cdot |F_2F_3| = 6 \cdot 4 = 24$. Therefore $\text{area}(P) = 24 - 1 - 1 - 2 - 4 = 16$.

11. Let r be a real root of the equation $x^3 - x + 1 = 0$. Evaluate the expression $r^5 + r^4 + r^2 + \dfrac{1}{r}$.

Answer: 0.

We are given that $r^3 - r + 1 = 0$. Hence $r^3 = r - 1$. It follows that $r^5 = r^2(r-1)$, $r^4 = r(r-1)$ and $r^2 = (r-1)/r$. Then

$$r^5 + r^4 + r^2 + \frac{1}{r} = r^2(r-1) + r(r-1) + \frac{r-1}{r} + \frac{1}{r} = (r^3 - r^2) + (r^2 - r) + 1 = r^3 - r + 1 = 0.$$

12. Suppose S_1 and S_2 are two circles of radius 1 that touch each other at the point O. Let S be a circle centered at O and tangent to both S_1 and S_2. Let S_0 be a circle that touches S internally and touches S_1 and S_2 externally. Find the radius of S_0.

Answer: $\frac{2}{3}$.

Suppose a circle C_1 with center O_1 and radius r_1 touches a circle C_2 with center O_2 and radius r_2. Then $|O_1O_2| = r_1 + r_2$ if the circles touch externally, and $|O_1O_2| = r_1 - r_2$ if C_2 touches C_1 from within. Also, the point at which the circles touch lies on the line O_1O_2.

Let R be the radius of the circle S and r be the radius of S_0. Let A, B_1 and B_2 be the centers of the circles S_0, S_1 and S_2, respectively. Since the circles S_1 and S_2 are of the same radius, they touch externally. Hence $|B_1B_2| = 1 + 1 = 2$ and the point O lies on the segment B_1B_2. The circles S_1 and S_2 touch the circle S from within. Therefore $|OB_1| = R - 1$ so that $R = |OB_1| + 1 = 1 + 1 = 2$. Since the circle S_0 touches S_1 and S_2 externally, we have $|AB_1| = |AB_2| = 1 + r$. The circle S is touched from within, hence $|AO| = R - r = 2 - r$.

Since $|OB_1| = |OB_2| = 1$, the segment AO is the median of the triangle B_1AB_2. As the triangle is isosceles, $|AB_1| = |AB_2|$, AO is also the altitude. It follows that the triangle AOB_1 is right. By the Pythagorean Theorem, $|AB_1|^2 = |AO|^2 + |OB_1|^2$. Hence $(1+r)^2 = (2-r)^2 + 1^2$. This equation is simplified to $6r = 4$, which has a unique solution $r = 2/3$.

13. Let f be the function of a real variable given by the formula

$$f(x) = \frac{cx + c^2}{3x - 6},$$

where c is a real number. Determine all values of the parameter c for which the function f is invertible and, moreover, coincides with its inverse function on the intersection of their domains.

Answer: 6.

The domain of the function f consists of all real numbers except 2. For any $x \neq 2$ we have

$$f(x) = \frac{cx + c^2}{3x - 6} = \frac{c}{3} \cdot \frac{x + c}{x - 2} = \frac{c}{3}\left(1 + \frac{c + 2}{x - 2}\right).$$

If $c = 0$ or $c = -2$, then the function f is constant and hence not invertible. Now assume $c \neq 0$ and $c \neq -2$. Since the function $h(x) = (x - 2)^{-1}$ takes all real values except zero, it follows that the function f takes all real values except $c/3$.

Take any $x \neq 2$ and let $y = f(x)$. By the above $y \neq c/3$. We obtain

$$y = \frac{cx + c^2}{3x - 6} \implies (3x - 6)y = cx + c^2 \implies (3y - c)x = 6y + c^2 \implies x = \frac{6y + c^2}{3y - c}$$

(note that $3y - c \neq 0$ since $y \neq c/3$). It follows that the function f is invertible, the inverse function g is given by the formula

$$g(y) = \frac{6y + c^2}{3y - c},$$

and the domain of the inverse function consists of all real numbers except $c/3$. Comparing the formulas for f and g, it is easy to observe that the two functions coincide if $c = 6$ (moreover, their domains are the same as well). Conversely, suppose $f(x) = g(x)$ for some real number x (different from 2 and $c/3$). Then

$$\begin{aligned}
\frac{cx+c^2}{3x-6} = \frac{6x+c^2}{3x-c} \quad &\Longrightarrow \quad (cx+c^2)(3x-c) = (6x+c^2)(3x-6) \\
&\Longrightarrow \quad 3cx^2 + 2c^2x - c^3 = 18x^2 + (3c^2-36)x - 6c^2 \\
&\Longrightarrow \quad (3c-18)x^2 + (36-c^2)x + c^2(6-c) = 0 \\
&\Longrightarrow \quad (c-6)\big(3x^2 - (c+6)x - c^2\big) = 0.
\end{aligned}$$

The polynomial $p(x) = 3x^2 - (c+6)x - c^2$ has at most two roots. Hence if the functions f and g coincide at three or more points, we must have $c = 6$.

14. Let P be a regular triangular pyramid (that is, the base of P is an equilateral triangle and the apex is projected onto the center of the base). It is given that three edges of the pyramid have length 4 and the other three edges have length 7. Find the volume of P.

Answer: $\frac{4}{3}\sqrt{131}$.

Let A, B and C be vertices of the base of the pyramid P. We have $|AB| = |AC| = |BC|$. Let O be the center of the base. Then $|OA| = |OB| = |OC|$. Let X be the apex of P. Then XO is the altitude of the pyramid. Hence in each of the triangles XOA, XOB and XOC, the angle at the vertex O is right. Since XO is the common side of the three triangles and $|OA| = |OB| = |OC|$, the triangles are congruent. As a consequence, $|XA| = |XB| = |XC|$. Let $a = |AB|$, $b = |XA|$. Then three edges of the pyramid P have length a and another three edges have length b. Therefore one of the numbers a, b equals 4 and the other equals 7.

Let AH be the altitude of the triangle ABC. Since this triangle is equilateral, AH is also the median, that is, $|BH| = |CH|$. Then OH is the median of the triangle BOC. The latter is isosceles, $|OB| = |OC|$. Hence OH is also the altitude. It follows that the point O lies on the segment AH. Note that AH is also the angle bisector. Hence $\angle OAB = \frac{1}{2}\angle CAB = \frac{1}{2}\cdot 60^\circ = 30^\circ$. Since the triangles AOB and BOC are clearly congruent, we have $\angle OBH = \angle OAB = 30^\circ$. From the right triangle OHB we obtain that $|HB| = |OB|\cos\angle OBH = |OB|\cos 30^\circ = |OB|\cdot\sqrt{3}/2$ and $|OH| = |OB|\sin\angle OBH = |OB|\sin 30^\circ = |OB|/2$. Therefore $|OB| = 2|HB|/\sqrt{3} = |BC|/\sqrt{3} = a/\sqrt{3}$. Then $|OH| = a/(2\sqrt{3})$, $|OA| = |OB| = a/\sqrt{3}$ and $|AH| = |OA| + |OH| = 3a/(2\sqrt{3}) = a\sqrt{3}/2$.

From the right triangle XOA we obtain that $|XA| > |OA|$. That is, $b > a/\sqrt{3}$. Recall that one of the numbers a, b equals 4 and the other equals 7. Since $7/\sqrt{3} = \sqrt{49/3} > \sqrt{48/3} = \sqrt{16} = 4$, it follows that $a = 4$ and $b = 7$.

The volume of the pyramid P equals $Sh/3$, where S is the area of the base and h is the length of the altitude. We have

$$S = \frac{1}{2}\,|BC|\cdot|AH| = \frac{1}{2}\,a\cdot\frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{4}.$$

Applying the Pythagorean Theorem to the right triangle XOA, we obtain $|XA|^2 = |XO|^2 + |OA|^2$. That is, $b^2 = h^2 + a^2/3$. Then $h = \sqrt{b^2 - a^2/3}$. Finally, the volume of P equals

$$\frac{1}{3}\cdot\frac{a^2\sqrt{3}}{4}\sqrt{b^2 - \frac{a^2}{3}} = \frac{a^2}{12}\sqrt{3b^2 - a^2} = \frac{4^2}{12}\sqrt{3\cdot 7^2 - 4^2} = \frac{4}{3}\sqrt{131}.$$

15. Find a triple of integers (a,b,c) such that $90 < a < b < c < 180$ and the sum of any two of the numbers a,b,c is a perfect square.

Answer: $(96, 129, 160)$. $[96+129 = 15^2,\ 96+160 = 16^2,\ 129+160 = 17^2.]$

Suppose $a+b = n_1^2$, $a+c = n_2^2$ and $b+c = n_3^2$, where a,b,c,n_1,n_2,n_3 are positive integers and $90 < a < b < c < 180$. Since $a < b < c$, it follows that $n_1 < n_2 < n_3$. Note that $n_1^2+n_2^2+n_3^2 = 2(a+b+c)$, which is an even number. Therefore either the numbers n_1, n_2 and n_3 are all even, or else one of them is even and the other two are odd. Besides,

$$a+b+c = \frac{n_1^2+n_2^2+n_3^2}{2},$$

which implies that

$$a = \frac{n_1^2+n_2^2-n_3^2}{2}, \qquad b = \frac{n_1^2+n_3^2-n_2^2}{2}, \qquad c = \frac{n_2^2+n_3^2-n_1^2}{2}.$$

Since $90 < a < b < c < 180$, it follows that $n_1^2 = a+b > 180 > 169 = 13^2$ so that $n_1 \geq 14$. Then $n_2 \geq 15$ and $n_3 \geq 16$. As for the upper bounds, $n_3^2 = b+c < 360 < 361 = 19^2$ so that $n_3 \leq 18$. Then $n_2 \leq 17$ and $n_1 \leq 16$. Further, $n_3^2 - n_1^2 = (b+c)-(a+b) = c-a < 180-90 = 90$. On the other hand, $n_3^2 - n_1^2 = (n_3+n_1)(n_3-n_1) \geq (16+14)(n_3-n_1) = 30(n_3-n_1)$. Hence $n_3 - n_1 < 90/30 = 3$, which implies that n_1, n_2 and n_3 are consecutive integers. Recall that one or three of them are even. This leaves only one possibility: $(n_1,n_2,n_3) = (15,16,17)$. As $15^2 = 225$, $16^2 = 256$ and $17^2 = 289$, we obtain

$$a = \frac{15^2+16^2-17^2}{2} = 96, \qquad b = \frac{15^2+17^2-16^2}{2} = 129, \qquad c = \frac{16^2+17^2-15^2}{2} = 160.$$

16. A regular dodecagon (12-gon) is inscribed into a circle of radius 1. How many diagonals of the dodecagon intersect the concentric circle of radius 1/3?

Answer: 18.

Let A and B be two vertices of the dodecagon and O be the center of the circle. If A and B are adjacent vertices, then AB is a side of the dodecagon and the angle $\angle AOB$ equals $2\pi/12 = \pi/6$. Otherwise AB is a diagonal and $\angle AOB = \pi k/6$, where $k = 2,3,4,5$ or 6. In the case $k = 6$, the diagonal AB goes through the center O and hence intersects any concentric circle. There are 6 such diagonals. For $2 \leq k \leq 5$, there are 12 diagonals such that $\angle AOB = \pi k/6$. In the latter case, the points A, B and O are vertices of a triangle. Let OH be the altitude of this triangle. Since the triangle is isosceles, $|OA| = |OB| = 1$, the altitude OH is also the angle bisector. The diagonal AB intersects the concentric circle of radius 1/3 if the distance from the diagonal to the center, which equals $|OH|$, is less than 1/3. Since OHA is a right triangle, $\angle AHO = \pi/2$, we obtain that $|OH| = |OA| \cos\angle AOH = \cos\angle AOH = \cos\frac{1}{2}\angle AOB = \cos(\pi k/12)$.

Now we need to compare the numbers $\cos(\pi/6)$, $\cos(\pi/4)$, $\cos(\pi/3)$ and $\cos(5\pi/12)$ with 1/3. The first three are greater than 1/3 as $\cos(\pi/6) > \cos(\pi/4) > \cos(\pi/3) = 1/2$. The last comparison is not so easy. Let $\alpha = \arccos(1/3)$ and $\beta = 5\pi/12$. The angles α and β lie between 0 and $\pi/2$. Using the formula $\cos(2\alpha) = 2\cos^2\alpha - 1$, we obtain $\cos(2\alpha) = 2(1/3)^2 - 1 = -7/9$. Besides, $\cos(2\beta) = \cos(5\pi/6) = -\sqrt{3}/2$. Note that both 2α and 2β lie in the range from 0 and π where the cosine function is decreasing. Since $-\sqrt{3}/2 = -\sqrt{60/80} < -\sqrt{49/81} = -7/9$, it follows that

$2\beta > 2\alpha$. Then $\beta > \alpha$ so that $\cos\beta < \cos\alpha = 1/3$. Thus we have 12 more diagonals that intersect the concentric circle of radius 1/3.

17. Find the sum of all even integers n in the range from 1 to 400 such that the sum $1 + 2 + 3 + \cdots + n$ is a perfect square.

Answer: 296. [There are two such integers, 8 and 288, $1 + 2 + \cdots + 8 = 6^2$, $1 + 2 + \cdots + 288 = 204^2$.]

Since $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$, we are looking for integer solutions of the equation $n(n+1) = 2k^2$ such that n is even. Notice that the numbers n and $n+1$ have no common divisors other than 1. Indeed, any common divisor of n and $n+1$ will also divide their difference $(n+1)-n = 1$. It follows that one of these numbers equals $2s_1^2$ and the other equals s_2^2, where s_1 and s_2 are integers with no common divisors other than 1, s_2 is odd, and $k = s_1 s_2$. Since n is supposed to be even, we have $n = 2s_1^2$ and $n+1 = s_2^2$. Further, $s_2 = 2n_1 + 1$ for some positive integer n_1 ($n_1 > 0$ since $n + 1 \geq 3$). Then $n = (2n_1+1)^2 - 1 = 4n_1(n_1+1)$. In particular, n is divisible by 4. Since $n = 2s_1^2$, the number s_1 is even: $s_1 = 2k_1$, where k_1 is an integer. Now $4n_1(n_1+1) = n = 8k_1^2$ or, equivalently, $n_1(n_1+1) = 2k_1^2$. Therefore (n_1, k_1) is another solution of the original equation (though n_1 need not be even). In particular, $1+2+3+\cdots+n_1 = k_1^2$. It follows from the construction that $n_1 = (\sqrt{n+1} - 1)/2$. If $n \leq 400$ then $n_1 \leq (\sqrt{401} - 1)/2 < (\sqrt{441} - 1)/2 = 10$. Calculating the sum $1+2+3+\cdots+n_1$ for small values of n_1 (up to 9), we obtain that $n_1 = 1$ (then $k_1 = 1$) or $n_1 = 8$ (then $k_1 = 6$). The corresponding values of n are computed by the formula $n = 4n_1(n_1+1)$. We get that $n = 8$ (then $k = 6$) or $n = 288$ (then $k = 204$). So, their sum is equal to 296.

18. In a triangle ABC with $|AB| > |AC|$, the median AM, the angle bisector AD and the altitude AH divide the angle BAC into 4 equal parts. Find $\angle ABC$.

Answer: $\dfrac{\pi}{8}$.

The triangles AHB and AHC are right, with a common leg AH. The inequality $|AB| > |AC|$ implies that $|BH| > |CH|$ and $\angle BAH > \angle CAH$. Since $|BM| = |CM|$ and $\angle BAD = \angle CAD$, it follows that points M and D lie between B and H. Further, the angle bisector AD divides the side BC of the triangle ABC into two segments whose lengths are proportional to lengths of the other two sides: $|BD|/|AB| = |CD|/|AC|$. In particular, $|BD| > |CD|$, which implies that the point M lies between B and D. Therefore the segments AM, AD and AH divide the angle BAC into 4 angles BAM, MAD, DAH and HAC.

Let α be the value of the equal angles BAM, MAD, DAH and HAC. Then $\angle ABC = \pi/2 - \angle BAH = \pi/2 - 3\alpha$ and $\angle ACB = \pi/2 - \angle CAH = \pi/2 - \alpha$. Applying the Law of Sines to the triangle BAM, we obtain $|BM|/\sin\alpha = |AM|/\sin(\pi/2 - 3\alpha)$. Applying the same law to the triangle CAM, we obtain $|CM|/\sin(3\alpha) = |AM|/\sin(\pi/2 - \alpha)$. Since $|BM| = |CM|$, it follows that

$$\frac{\sin(\pi/2 - 3\alpha)}{\sin\alpha} = \frac{\sin(\pi/2 - \alpha)}{\sin(3\alpha)}.$$

Then $\sin(3\alpha)\sin(\pi/2-3\alpha) = \sin\alpha\sin(\pi/2-\alpha)$ or, equivalently, $\sin(3\alpha)\cos(3\alpha) = \sin\alpha\cos\alpha$. Hence $\sin(6\alpha) = \sin(2\alpha)$. Note that $4\alpha = \angle BAC < \pi$. Therefore $0 < 2\alpha < \pi/2$ and $0 < 6\alpha < 3\pi/2$. Clearly, $2\alpha < 6\alpha$. It follows that $6\alpha = \pi - 2\alpha$. Then $\alpha = \pi/8$ and $\angle ABC = \pi/2 - 3\alpha = \pi/8$.

19. The number $5 \cdot 3^2 \cdot 2^{336}$ is the smallest positive integer that has exactly 2022 different divisors. How many digits does this number have when written out (in decimal notation)?

Answer: 103.

The number $N = 5 \cdot 3^2 \cdot 2^{336}$ has d digits when written out if $10^{d-1} \le N < 10^d$. Our first observation is that the number $2^{10} = 1024$ is close to $10^3 = 1000$. We have

$$N = 5 \cdot 3^2 \cdot 2^{-4} \cdot (2^{10})^{34} = \frac{45}{16} \cdot (2^{10} \cdot 10^{-3})^{34} \cdot (10^3)^{34} = \frac{45}{16} \cdot \left(\frac{1024}{1000}\right)^{34} \cdot 10^{102}.$$

Clearly, $2 < 45/16 < 3$. Further,

$$1 < \left(\frac{1024}{1000}\right)^{34} < \left(\frac{1025}{1000}\right)^{34} = \left(\frac{41}{40}\right)^{34} = \left(1 + \frac{1}{40}\right)^{34}.$$

By the binomial formula,

$$\left(1 + \frac{1}{40}\right)^{34} = \sum_{k=0}^{34} \frac{34!}{k!\,(34-k)!} \left(\frac{1}{40}\right)^k.$$

The first two terms in the sum (for $k = 0$ and $k = 1$) are equal to 1 and 34/40. For $k \ge 2$ we get

$$\frac{34!}{k!\,(34-k)!} \left(\frac{1}{40}\right)^k = \frac{1}{2 \cdot 3 \cdot \ldots \cdot k} \cdot \frac{34}{40} \cdot \frac{33}{40} \cdot \ldots \cdot \frac{35-k}{40} < \frac{1}{2 \cdot 3 \cdot \ldots \cdot k} \le \frac{1}{2^{k-1}}.$$

Therefore

$$\left(1 + \frac{1}{40}\right)^{34} < 1 + \frac{34}{40} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{33}} = 1 + \frac{34}{40} + 1 - \frac{1}{2^{33}} < 3.$$

Now it follows from the above that $2 \cdot 10^{102} < N < 9 \cdot 10^{102}$. Thus $d = 103$.

20. Find the least positive integer n such that the sum

$$\sin 14^\circ + \sin 28^\circ + \sin 42^\circ + \cdots + \sin(14n)^\circ$$

has negative value.

Answer: 25.

To turn this sum into a telescopic one, we multiply and divide each term by $2\sin 7^\circ$. Trigonometric formulas $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ imply that $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$. Applying the last formula, we obtain for any integer $k \ge 1$ that

$$2\sin(14k)^\circ \sin 7^\circ = \cos(14k-7)^\circ - \cos(14k+7)^\circ.$$

Therefore

$$\sin 14^\circ + \sin 28^\circ + \cdots + \sin(14n)^\circ = \frac{2\sin 14^\circ \sin 7^\circ + 2\sin 28^\circ \sin 7^\circ + \cdots + 2\sin(14n)^\circ \sin 7^\circ}{2\sin 7^\circ}$$

$$= \frac{(\cos 7^\circ - \cos 21^\circ) + (\cos 21^\circ - \cos 35^\circ) + \cdots + \big(\cos(14n-7)^\circ - \cos(14n+7)^\circ\big)}{2\sin 7^\circ}$$

$$= \frac{\cos 7^\circ - \cos(14n+7)^\circ}{2\sin 7^\circ}.$$

Applying the same trigonometric formula again, with $\alpha = (7 + 7n)^\circ$ and $\beta = (7n)^\circ$, we obtain

$$\sin 14^\circ + \sin 28^\circ + \cdots + \sin(14n)^\circ = \frac{2\sin(7n+7)^\circ \sin(7n)^\circ}{2\sin 7^\circ} = \frac{\sin(7n+7)^\circ \sin(7n)^\circ}{\sin 7^\circ}.$$

Since $\sin 7^\circ > 0$, the value of the sum is negative when one of the numbers $\sin(7n)^\circ$ and $\sin(7n+7)^\circ$ is positive while the other is negative. Note that $180 = 25 \cdot 7 + 5$. Hence for any n satisfying $1 \le n \le 24$ we have $0 < 7n < 7n + 7 < 180$ so that both sines are positive. For $n = 25$, we have $0 < 7n < 180 < 7n + 7 < 360$ so that $\sin(7n)^\circ > 0$ while $\sin(7n + 7)^\circ < 0$.