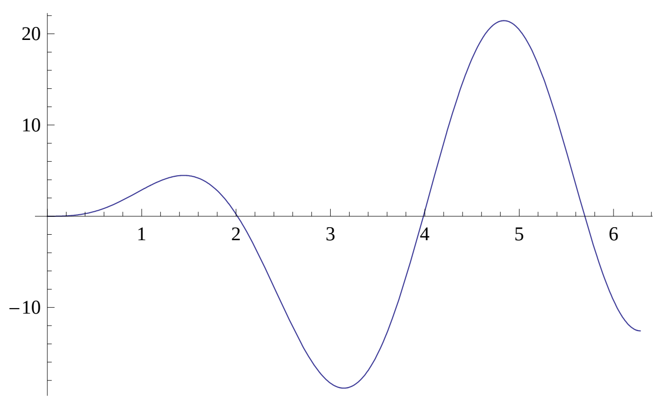
TAMU 2008 Freshman-Sophomore Math Contest

First-year student version

There are five problems, each worth 20% of your total score. This is not an examination, and a good score, even a winning score, can be well short of solving all five problems completely. See what you can do with these. Rules: No laptop computers, no calculators, no cell phones or other means of communicating with the outside. You're on your own for the duration of the contest. Blank paper and pencils are provided.

1. Show that
$$\int_{u=0}^{2\pi} \frac{\sin u}{u}\,du = \pi \int_{u=0}^{\pi} \frac{\sin u}{u(\pi+u)}\,du.$$
Let A be the integral on the left, and B, the integral on the right. We have
$$A = \int_0^{2\pi} \frac{\sin u}{u}\,du = \int_0^{\pi} \frac{\sin u}{u}\,du + \int_{\pi}^{2\pi} \frac{\sin u}{u}\,du.$$
The second integral is, by a change of variable replacing u with $u-\pi$, equal to $\int_0^\pi \sin(u+\pi)/(u+\pi)\,du$. Now $\sin(u+\pi) = \sin(u)\cos(\pi)+\sin(\pi)\cos(u) = -\sin(u)$. Thus
$$A = \int_0^{\pi} \sin u \left(\frac{1}{u} - \frac{1}{u+\pi}\right) du = \int_0^{\pi} \sin u \left(\frac{\pi+u}{u(u+\pi)} - \frac{u}{u+\pi}\right) du = B.$$

2. Let
$$F(x) = \int_{x^2}^{4x^2} \sin(\sqrt{t})\,dt.$$



(Graph of $F(x)$)

   (a) Find all critical points of F between 0 and 2π. There are two natural ways to go at this. One would be to evaluate the integral, (substitution, $t = s^2$), then take the derivative, then work out what values of

x give a derivative of zero. The other, which we choose here, would be to find the derivative without doing the integral, with the rest of the plan being the same.

The fundamental theorem of calculus says that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ if f is continuous. Here, our integral has upper and lower limits that are both variable, and they're both functions of x, rather than simply x. No matter. We use the chain rule. Thus,

$$\frac{d}{dx}\int_{x^2}^{4x^2} f(t)\,dt = 8xf(4x^2) - 2xf(x^2).$$

Since here, $f(x) = \sin\sqrt{x}$, we have $F'(x) = 8x\sin(2x) - 2x\sin(x)$. Clearly this is zero at $x = 0, \pi$, and 2π. But from the graph, there must be other points as well.

With the trigonometric identity $\sin 2x = 2\sin x\cos x$, we have $F'(x) = 2x\sin x(8\cos x - 1)$. Now the other two critical points pop into focus: they're the places where $\cos x = 1/8$, and those are $\arccos(1/8)$ and $2\pi - \arccos(1/8)$.

Remark: It may seem as though this solution merely kicks the ball down the road, because we still don't have a 'real' answer, like '7' or '$\sqrt{2}$'. But the latter isn't any more of a 'real' solution than $\arccos(1/8)$. Both numbers are defined as inputs to a function that will give a desired, simple output. If we want decimal digits for $\sqrt{2}$, or for $\arccos(1/8)$, we will need something such as Newton's method, applied to $x^2 - 2$ or $\cos x - 1/8$, and some patience or a computer, to pin down the details. For the record, $\arccos(1/8)$ is about 1.4454684956268312224.

(b) Evaluate the integral and find a closed-form expression for $F(x)$. We need an indefinite integral for $u\sin u$. Integration by parts is just the trick, and it gives $\sin u - u\cos u = \int u\sin u\,du$. So,

$$\begin{aligned}\int 8x\sin 2x\,dx &= 2\int u\sin u\,du = 2(\sin u - u\cos u)\\ &= 2(\sin 2x - 2x\cos 2x)\end{aligned}$$

and $\int 2x\sin x = -2x\cos x + 2\sin x$, and this gives $F(x) = 2\sin 2x - 4x\cos 2x + 2x\cos x - 2\sin x$. What about '+C'? We're OK here because $F(0) = 0$ from its definition, just like our answer gives.

3. Let

$$g(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{2^{n^2}}.$$

Here, the denominator is 2 to the power n^2.

(a) Show that $g(2) > 0$, $g(4) > 0$, and $g(8) > 0$. One thing to keep in mind is that an alternating series with terms that decrease in absolute value, and decrease to zero, converges, and that such a sum converges to a value between 0 and its first term.

Our sum for $g(x)$ is not like that, because the terms initially increase in absolute value. However, we can strip off any finite number of starting terms, and add those up separately, and then use this observation on the tail of the series. So, let $a(x)$ be the sum of all the terms with absolute value 1 or more, and $b(x)$, the rest.

We have $a(2) = 1$, while $b(2) = -2^{-2} + 2^{-6} - 2^{-12} \cdots < 0$. Thus, $g(2) < 0$. For 4, we have $a(4) = 1 - 1 = 0$ and $b(4) = 2^{-3} - 2^{-5} + 2^{-15} \cdots > 0$. For 8, we have $a(8) = 4 - 4 + 1 > 0$ and $0 < b(8) < 1$, so $g(8) > 0$.

(b) Show that $g(16) < 0$. We have

$$\begin{aligned} b(16) &= \sum_{n=5}^{\infty}(-1)^{n-1}(16^n)/2^{n^2} \\ &= 16^5/2^{25} - 16^6/2^{36} + \cdots = 2^{-5} - 2^{-12} + 2^{-21} \cdots \end{aligned}$$

so that $0 < b(16) < 1/32$. As to $a(16)$, it is $2^3 - 2^4 + 2^3 - 1 = -1$. With $a(16) = -1$ and $b(16) < 1/32$, it follows that $g(16) < 0$.

(c) Show that $g(2^8) < 0$. This will be subsumed into the last part.

(d) Generalize. That is, state when $g(2^k)$ is positive, and explain. $g(2^k)$ is negative when k is a multiple of 4, and positive otherwise.

What seems to be happening is that the main stuff, $a(x)$, consists of a sum which is the same read forward and backward, apart from the business of the alternating signs. This suggests rearranging the sum so that its central term is the zeroth term, and the other terms are the $\pm l$ terms, l running from minus somewhere, to plus the same.

When $k = 4m$, we have

$$\begin{aligned} a(2^k) &= 2^{4m-1} - 2^{8m-4} + 2^{12m-9} + \cdots + (-1)^{4m-1}2^{16m^2-(4m)^2} \\ &= \sum_{j=1}^{4m}(-1)^{j-1}2^{4mj-j^2} = \sum_{l=-2m+1}^{2m}(-1)^{l-1}2^{4m^2-l^2}. \end{aligned}$$

The term $l = 0$ is largest, and gives a negative value because of the factor $(-1)^{l-1}$. The terms $l = \pm 1$, taken together, cancel that out. So, we are back to zero. But now the terms $l = \pm 2$ give a negative contribution, and the terms $l = \pm 3$, if the sum goes that far, give a smaller positive contribution, and so on out to the end. The terms $l = \pm 2$ give $2 \cdot 2^{4m^2-4}$ which is an even positive integer. As we observed at the outset, and alternating sum with terms that decrease in absolute value yields a total with the same sign as the first term. Thus, we get a negative integer for $a(2^{4m})$. As usual, $|b(x)| < 1$, so

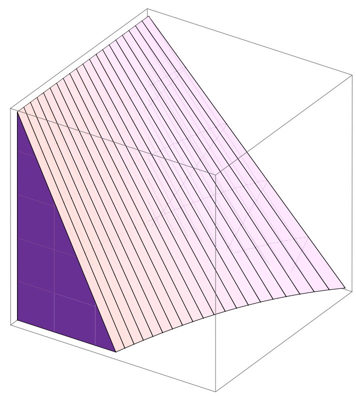
$g(2^{4m}) < 0$. We sketch the rest of the cases. If $k = 4m+1$, we have

$$\begin{aligned} a(2^k) &= \sum_{j=1}^{4m+1} (-1)^j 2^{(4m+1)j-j^2} = -2^{4m^2+6m} \sum_{l=-2m+1}^{2m+1} (-1)^l 2^{l-l^2} \\ &= -2^{4m^2+6m} \sum_{l=1}^{2m+1} \left[(-1)^l 2^{l-l^2} + (-1)^{1-l} 2^{(1-l)-(1-l)^2} \right] \\ &= -2^{4m^2+6m} \sum_{l=1}^{2m+1} (-1)^l \left[2^{l-l^2} - 2^{l-l^2} \right] = 0. \end{aligned}$$

The main terms having canceled, the first term of $b(x)$ rules, and that is the $k = 4m+2$ term, which is positive. So $g(2^{4m+1}) > 0$. For $k = 4m+2$, the situation is just like it is for $k = 4m$, with a central term canceled by its flanking pair of terms. But then, the largest pair of surviving near-central terms, corresponding to $k = 2m-1$ and $k = 2m+3$, gives a positive contribution. Thus $g(2^{4m+2}) > 0$. For $k = 4m+3$, the terms pair off, with the middle pair being terms number $2m+1$ and $2m+2$, except for the first term, which remains unpaired. Thus $a = 1$, and as usual, b is too small to change the verdict. $g(2^{4m+3}) > 0$.

4. A girl walks along the edge of a 100 meter high cliff, going North at 1 meter per second, along a path 100 meters in length. Below, a ship keeps even with her, but offshore (West), a distance of $10000/(100+t)$ meters at time t, starting at $t = 0$. Thus, initially, the line-of-sight distance from girl to ship is $\sqrt{20000}$ meters. After 100 seconds, they're both 100 meters North of where they started, but the ship is only 50 meters clear of the rocks, and the line-of-sight distance between girl and ship has diminished to $\sqrt{12500}$ meters.

What is the average value of the distance from girl to ship over the duration of her stroll?



(view from a seagull above and to the northwest of the path) At time t, her North-South coordinates matches that of the ship. Her Up-Down coordinate differs by 100. Her East-West coordinate differs by $10000/(100+t)$, so the overall distance at time t is $\sqrt{10^4 + 10^8/(100+t)^2}$. Making first the substitution $100u = 100 + t$, and then the substitution $u = \tan\theta$, the average distance A is

$$
\begin{aligned}
A &= \frac{1}{100}\int_{t=0}^{100} \sqrt{10^4 + \frac{10^8}{(100+t)^2}}\, dt = \int_{u=1}^{2} \sqrt{10^4 + \frac{10^4}{u^2}}\, du \\
&= 100 \int_1^2 \sqrt{\frac{u^2+1}{u^2}}\, du = 100 \int_{\theta=\arctan 1}^{\arctan 2} \frac{\sec\theta}{\tan\theta} \sec^2\theta\, d\theta \\
&= \int_{\arctan(1)}^{\arctan(2)} \sec^3\theta/\tan\theta\, d\theta = \int_a^b \csc\theta + \sec\theta\tan\theta \\
&= -\log(\csc\theta + \cot\theta) + \sec\theta\, \Big|_a^b ,
\end{aligned}
$$

where $a = \arctan 1 = \pi/4$, and $b = \arctan(2)$. Now in view of $\cot\arctan x = 1/x$, $\sec\arctan x = \sqrt{1+x^2}$, and $\csc\arctan x = \sqrt{1+x^2}/x$, this gives an average distance of

$$
\begin{aligned}
&100\left(-\log(1+\sqrt{1+x^2}) + \log x + \sqrt{1+x^2}\,\Big|_1^2\right) \\
&= 100\left((-\log(1+\sqrt{5}) + \log 2 + \sqrt{5}) - (-\log(1+\sqrt{2}) + \sqrt{2})\right).
\end{aligned}
$$

Further whimsical question: is the girl's name Lorelei? Why, or why not?

5. The region shown is bounded by the lines from the ship to the girl as she walked the path, by lines from the shore at the base of the cliff below the girl, to the ship, by the face of the cliff itself for that 100 meter stretch, and by the triangles at either end of the region. Find the volume of the region (in cubic meters.) The volume can be calculated several ways. Each of the up-down/east-west cross sections is a triangle. The triangle at distance t along the path has height 100, and base of length $10000/(100+t)$. Thus the volume is the integral of the triangle cross-section areas, and that is

$$5 \cdot 10^5 \int_0^{100} \frac{1}{100+t}\,dt = 500\,000 \ln(2).$$