TAMU 2009 Freshman-Sophomore Math Contest

Second-year student version

There are five problems, each worth 20% of your total score. This is not an examination, and a good score, even a winning score, can be well short of solving all five problems completely. See what you can do with these. Rules: no aids to calculation, no cell phones or other means of communicating with the outside world. You're on your own for the duration of the contest. Blank paper and pencils are provided.

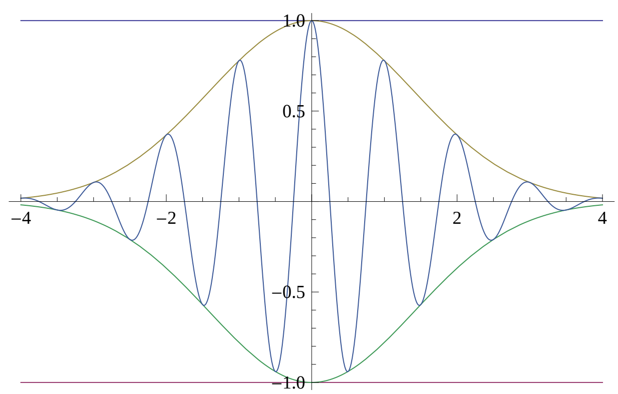
1. Consider the function $\phi(x)$ given by

$$\phi(x) = e^{-x^2/4}\cos(2\pi x).$$

(a) Find $\phi'(x)$. That would be

$$\begin{aligned} &-(1/2)e^{-x^2/4}x\cos(2\pi x) - 2e^{-x^2/4}\pi\sin(2\pi x) = \\ &-(1/2)e^{-x^2/4}\left(x\cos(2\pi x) + 4\pi\sin(2\pi x)\right). \end{aligned}$$

(b) Sketch the graph of $y = \phi(x)$ for the interval $-4 \le x \le 4$. The graph of $e^{-x^2/4}$ is supplied; your sketch goes into the same plot window. [Here's the computer-generated sketch.]



(c) The minimum value of $\phi(x)$ occurs at an x-value somewhere between $1/4$ and $3/4$. Is this value less than $1/2$, equal to $1/2$, or greater than $1/2$? Explain. Ideally, prove your answer. The minimum (it's global) occurs somewhere that $\phi'(x) = 0$. The first factor in our expression for $\phi'(x)$ is never zero, so that place must be somewhere that the other factor is zero. Simplifying leads to this condition: $-x/(4\pi) = \tan(2\pi x)$. Since the left hand side of this equation is negative when $x > 0$, the right hand side must also be negative. That means $2\pi x$, the thing whose tangent is taken, is in the interval $(\pi/2, \pi)$, so x itself is in the interval $(1/4, 1/2)$. It's less than $1/2$.

2. Let

$$A = \int_{x=0}^{\infty} \frac{\sin x}{x}\, dx, \quad B = \int_{0}^{\infty} \frac{\sin(x^2)}{x}\, dx.$$

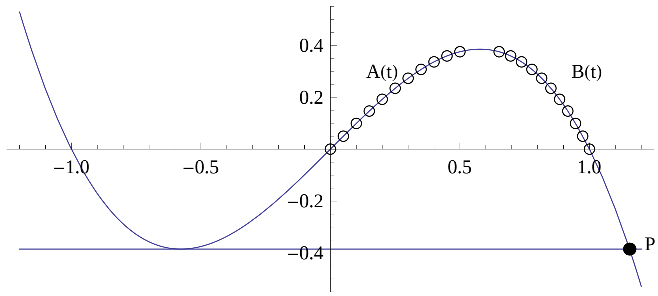
(a) Both integrals converge. Why? One answer is that the sine wave alternates, so the region between the curve and the x-axis is positive, then negative but less so, then positive but still less, and so on. This means that we have an alternating series of terms that decrease in absolute value, and such series converge. Another answer involves integration by parts. If you take $U = 1/x$, and dV to be the sine wave, then $dU = -1/x^2$, while V is either $-\cos(x)$, which is at any rate no more than 1, or something else that is bounded. The reason it's bounded is that the integral of $\sin(x^2)$ is another of those alternating series situations.

(b) Nothing in your studies up to now is likely to have equipped you to find either A or B. Nevertheless, it is possible to find A/B without any methods beyond the scope of introductory calculus. Find A/B.

In the integral defining B, make the change of variable $x^2 = u$. Equivalently, $x = u^{1/2}$, and then $dx = (1/2)u^{-1/2}$. That gives

$$\begin{aligned} B &= \int_{u=0}^{\infty} u^{-1/2} \sin(u) \cdot (1/2)u^{-1/2}\, du \\ &= \frac{1}{2} \int_{u=0}^{\infty} \frac{\sin(u)}{u}\, du = \frac{1}{2} A. \end{aligned}$$

3. Let $g(x) = x - x^3$. The graph of $g(x)$, together with other features of this problem, is shown on the interval $[-1.2, 1.2]$.



(a) Find the (exact) value of the point P at which the tangent line to the curve at its local minimum intersects the curve again.

In the spirit of calculus, let's take the derivative and set it to zero to find out where that local minimum is. We have $g'(x) = 1 - 3x^2$, and this is zero at $x = -1/\sqrt{3}$. There, $g(x) = -2/(3\sqrt{3})$. Now we have to solve $x - x^3 = -2/(3\sqrt{3})$ and find the positive value of x satisfying that equation. Let $b = -2/(3\sqrt{3})$ and $a = -1/\sqrt{3}$.

Solving cubics is no fun in general, but here we have a bit of help. Finding x so that $x - x^3 = b$ is the same as finding x so that $x^3 - x + b = 0$. We know that $x = a$ is one solution. So divide $(x - a)$ into $x^3 - x + b$ and arrive at $x^2 + ax - 2/3 = 0$. This is a quadratic and with routine calculation you get $x = a$ (again), or $x = -2a = 2/\sqrt{3}$. The required point is $(2/\sqrt{3}, -2/(3\sqrt{3}))$.

(b) Show that if u and v are distinct real numbers so that $g(u) = g(v)$ then $u^2 + uv + v^2 = 1$.

If $g(u) = g(v)$ then $u^3 - u = v^3 - v$ so $u^3 - v^3 = u - v$. Dividing by $u - v$ is OK so long as $u \neq v$, and then $u^2 + uv + v^2 = 1$.

(c) Point A starts at $(0, 0)$ at time $t = 0$ and moves up and right along the graph of $y = g(x)$. Point B starts at $(1, 0)$ and moves up and left along the graph of $y = g(x)$ so as to be at the same height as A. If the x-coordinate of A is increasing at rate 3 when $t = 0$, at what rate is the x-coordinate of B decreasing when $t = 0$? The slope on the left is 1. The slope on the right is -2. If the two points are going to keep pace in their y coordinates, then A has to move to the right twice as fast as B moves left. The x coordinate of B is decreasing at rate $3/2$.

4. Starting from a randomly chosen * point somewhere in the northern hemisphere, an airplane flies to the North pole. What is the average distance the plane will have to fly? Take the radius of the earth to be 3000 miles.

* (A point taken at random is as likely to fall within any square mile patch of the northern hemisphere as any other.)

An area integral over the top hemisphere is called for, and then to get the average, divide by $2\pi\rho^2$.The area element in spherical coordinates is $\rho^2 \sin\phi \, d\phi \, d\theta$. The area of a sphere is $4\phi\rho^2$, so we'll be dividing our integral by $2\pi\rho^2$, where $\rho = 3000$. Now the distance, flying along the great circle route to the north pole, of a point with spherical coordinates (ρ, ϕ, θ) is $\rho(\pi/2 - \phi)$.

The appropriate integral is thus

$$D = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \rho(\pi/2 - \phi)\rho^2 \sin\phi \, d\phi \, d\theta = 2\pi\rho^3 \int_{\phi=0}^{\pi/2} (\pi/2 - \phi) \sin\phi \, d\phi.$$

Thus the average distance is given by $3000 \int_{\phi=0}^{\pi/2} (\pi/2 - \phi) \sin\phi \, d\phi$.The first term in this last integral is just $\pi/2$. For the second term we need integration by parts. Taking $U = \phi$, $dV = \sin\phi \, d\phi$ gives $-\phi \cos\phi|_0^{\pi/2} + \int_0^{\pi/2} \cos\phi \, d\phi = 0 + 1 = 1$. So the integral is $\pi/2 - 1$ and the average distance to fly is $3000(\pi/2 - 1)$ (miles).

5. A rainstorm blows through, with rain falling at the rate of $12(t - t^2)$ inches per hour over the course of one hour. The rate at which it drains off the sidewalk is proportional to the depth of the accumulated water; when the water is one inch deep it's draining at the rate of one inch per hour. How many inches deep is the water on the sidewalk at the end of the one-hour storm?

   Let $W(t)$ be the number of inches water accumulated at time t, measuring t in hours and water in inches. The differential equation governing the situation is then $W' = 12(t - t^2) - W$. This is first-order linear, and we proceed with $W' + W = 12(t - t^2)$, so the integrating factor is e^t and $(We^t)' = 12(t - t^2)e^t$, with initial condition $W(0) = 0$. Integration by parts gives $We^t = 12(-t^2 + 3t - 3)e^t - C$ and with the initial condition,

$$W = 12(-t^2 + 3t - 3) + 36e^{-t}.$$

   The water is $36e^{-1} - 12$ inches deep at the end of the storm, or 1 and a quarter inches, roughly. Two inches fell.