TAMU 2013 Freshman-Sophomore Math Contest

Solutions Freshman Version

While the name of the contest is traditional, the actual eligibility rules are that first year students take the freshman contest, and second year students take the sophomore contest. That way, students who have accummulated enough credit hours in their first or second year to have standing as sophomores, or juniors, are not promoted out of eligibility.

The first page contains problems built around Calculus I and II for both freshmen and sophomores. The second pages are pitched to content unique to Calculus III and/or Differential Equations, in the case of the sophomore contest.

In all cases, solutions should be written out and should include reasoning behind the steps when reasons beyond routine calculation are involved. No tables, calculators, or computers, and no devices for communication with the outside world, are allowed. You're on your own.

1. Find

$$\lim_{x\to\infty} \frac{\ln\left(1+e^{2x}+e^{4x}\right)}{\ln\left(6+e^{8x}+e^{10x}\right)}.$$

Solution: Observe that $e^{ax}+e^{bx}$ is dominated by the term associated with the larger of a and b. Thus $\lim_{x\to\infty}(1+e^{2x}+e^{4x})/e^{4x}=1$, and likewise, $(6+e^{8x}+e^{10x})/e^{10x}$ tends to 1. Thus the logs of these ratios tend to zero, or equivalently, the difference between $\ln(6+e^{8x}+e^{10x})$ and $\ln e^{10x}$ tends to zero.

Our numerator thus amounts to $4x+$ stuff that goes to zero, and the denominator, to $10x+$ stuff that goes to zero, so the limit is 2/5 and that is our answer.

2. Let $f(x)=\tan(\ln(\cos x+\sin x))$ (where defined—there will be points at which the definition of f breaks down).

   (a) Find a formula for $f'(x)$ which holds where f is defined.
   Solution: By the chain rule, the derivative is

$$f'(x)=(\cos x-\sin x)\cdot\frac{1}{\cos x+\sin x}\cdot\sec^2\ln(\cos x+\sin x).$$

   (b) Find the number nearest 0 at which f is not defined. There are two things that can cause f to not be defined. First, the expression whose log is required may be zero or negative. This happens when $\sin x=-\cos x$, or equivalently, when $\tan x=-1$. This occurs at $-\pi/4$ and at $3\pi/4$ and in general at $-\pi/4+k\pi$ where k is an integer. The nearest of these is $-\pi/4$ itself.

Second, the log may exist, but its tangent not exist because the log is itself an odd integer multiple of $\pi/2$. Now, we must figure out where $\log(\cos x + \sin x) = \pi/2, -\pi/2, -3\pi/2$ and so forth. The first does not occur because $\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$ by trigonometry, and $\sqrt{2} < \pi/2$. The second does occur, because all we need is that $\cos x + \sin x$ be near enough zero that the log is strongly negative.

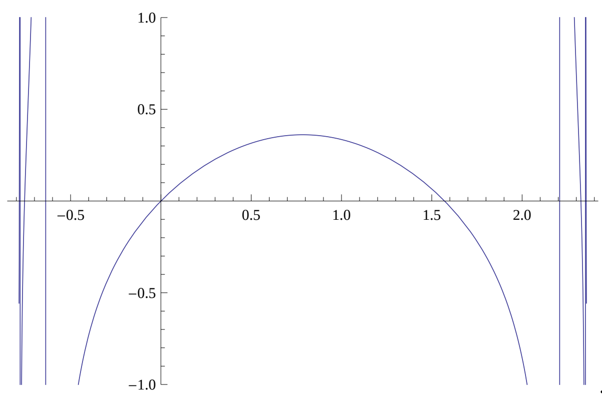
The nearest point will be where $\cos x + \sin x = e^{-\pi/2}$, or equivalently, where $\cos(x - \pi/4) = 2^{-1/2}e^{-\pi/2}$. So, $x - \pi/4 = -\arccos(2^{-1/2}e^{-\pi/2})$. Why the minus sign? Because traditionally, arccos is defined on $[0, \pi]$ and we want a negative number for $x - \pi/4$ because we know our answer is near $-\pi/4$ rather than near $3\pi/4$.

Finally, the answer: $x = \pi/4 - \arccos(2^{-1/2}e^{-\pi/2})$.

(c) Sketch the graph of $f(x)$ on the interval $(-\pi/4, 3\pi/4)$, indicating such discontinuities as may exist. How many are there, in all, on that interval? Why?

Solution: There are infinitely many points of discontinuity, one at each place where $\cos x + \sin x$ hits a small number near enough zero that its log takes the form $-k/pi/2$. These will cram up against the endpoints of our interval $-/pi/4$ and $/pi/4$.

The graph will thus look like this in the middle, and zooming in, at the edges:



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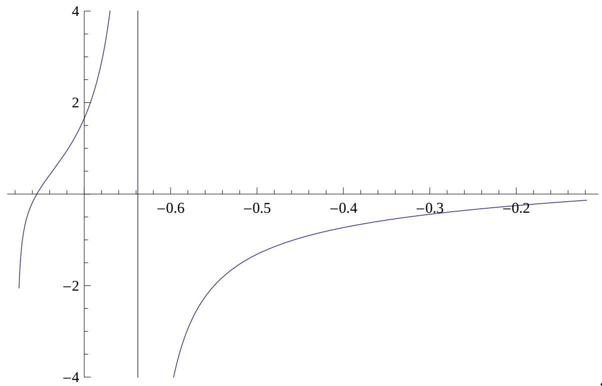

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3. Find, accurate to within ± 0.0001, the numerical value as a decimal of the form $a.bcd$ of

$$\int_{x=0}^{1} \frac{1}{x}\sin(x^2)\,dx.$$

Solution: this is a job for Ceres, Goddess of the Harvest! (Here, spelled Series.) The series expansion of $\sin z$ is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots .$$

(To the question, how am I supposed to know that?—the answer must be, by rote. Like the times table. It's a basic fact that crops up so frequently that knowing it is indispensable.) Anyhow, with some manipulation, $\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots$, and $(1/x)\sin x^2 = x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots$. Now this can be integrated term by term, with the result that the required definite integral is given by a rapidly converging, alternating sum with a value $A = \frac{1}{2} - \frac{1}{6\cdot 3!} + \frac{1}{10\cdot 5!} - \cdots$. We have enough terms already and it is time for some arithmetic: $\frac{1}{2} - \frac{1}{36} + \frac{1}{1200} = \frac{1800-100+3}{3600} = \frac{1703}{3600} = 0.4731$, rounded up to the nearest ten-thousandth. The actual value is nearer 0.473042, or if you want lots and lots of places,

$$A = 0.4730415351835915074706766569115898289061689773690558952357273867833435182703989 6$$

4. Let

$$A = \int_{x=0}^{1} \left(1 - \frac{1}{x^2}\right) e^{-(x+1/x)}\,dx.$$

(a) Prove that the improper integral defining A converges.

Solution: The improper integral is by definition the limit as R tends to zero from above of

$$\lim_{R\to 0^+} \int_{x=R}^{1} \left(1 - \frac{1}{x^2}\right) e^{-(x+1/x)}\,dx.$$

The integrand inside the limit is negative and greater than $-(1/x^2)e^{-1/x}$, so all we need is to show that with that for the new integrand, the expression converges. With the change of variable $y = 1/x$, our integral becomes $-\int_{y=1/R}^{1} e^{-y}\, dy$ which tends to a finite value. Which finite value is immaterial, but, for the record, it s $1/e - 1$. Thus the original integral is a convergent improper integral. (The key fact was that the $e^{-1/x}$ was tending to zero more strongly than the fraction $1/x^2$ was tending to infinity, at $x = 0$.)

(b) Evaluate A.

With the change of variable $u = x + 1/x$, we have $du = 1 - 1/x^2$ and our integral becomes $\int_{u=\infty}^{2} e^{-u}\, du$. Rearranging the limits of integration to the traditional lower first and flipping signs, we get $-e^{-2}$ for our answer. That is, $A = -e^{-2}$.

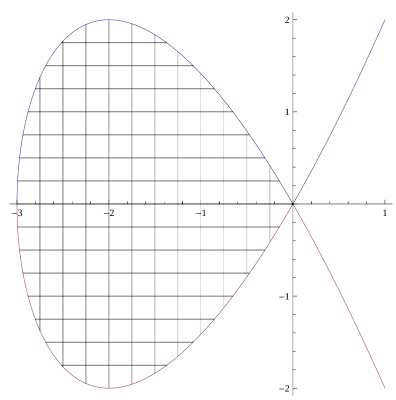
5. Determine, with proof, whether

$$\sum_{n=2}^{\infty} \frac{2^n}{n! - (n-1)!}$$

converges. Solution: First note that for $n \geq 2$, $n! = n(n-1)$ so that $n! \geq 2(n-1)!$. The denominator is thus at least $(1/2)n!$. Our sum is thus a sum of positive terms, yet less than $\sum_{n=2}^{\infty} 2^{n-1}/n!$. The ratio of $n+1$th term to nth term in this new sum is $2/(n+1)$ which tends to zero, so by the ratio test, the new and larger sum converges, and thus by the comparison test, the original sum converges.

6. Find the area of the finite region enclosed by the graph of the equation

$$y^2 = x^2(x+3).$$



Solution: The upper and lower bounding curves are $\pm - x\sqrt{x+3}$, as x goes from -3 to 0. Thus the area is $2\int_{x=-3}^{0} -x\sqrt{x+3}\, dx$. With the change of variable $u = -x$, this becomes $+2\int_{u=0}^{3} u\sqrt{3-u}\, du$, and on the general principle that difficulties should be spread out rather than piled all in one corner, we make the substitution $v = 3 - u$. This gives $2\int_{v=0}^{3}(3-v)\sqrt{v}\,dv = 2\int_{v=0}^{3} 3v^{1/2} - v^{3/2}\,dv = 24\sqrt{3}/5$.