TAMU 2015 Freshman-Sophomore Math Contest

Solutions Freshman Version

1. Let (X, Y) be the point nearest $(1, 1)$ that's below the parabola $y = x^2$ and at least 1 unit distant from every point on the parabola. Find $X + 2Y$.

   The answer is 3. To find (X, Y), we look for a point at distance 1 from $(1, 1)$ that is below the parabola (so $Y < X^2$) and that is not at a distance of 1 from any other point on the parabola. If we arrange that the circle of radius 1 about (X, Y) is tangent to the parabola at $(1, 1)$, that will happen. The slope of the tangent line to the parabola there is 2, so we want the slope of the radius from $(1, 1)$ to (X, Y) to be $-1/2$. That means $X = 1 + t$, $Y = 1 - t/2$, with t chosen so that $t\sqrt{1 + 1/4} = 1$. So $t = 2/\sqrt{5}$, $X = 1 + 2/\sqrt{5}$, $Y = 1 - 1/\sqrt{5}$, and $X + 2Y = 3$.

2. Let
$$f(x) = \frac{\log x}{1 + x^2}, \quad L = \int_1^\infty f(x)\, dx.$$

   (a) Prove that L is finite.

   The answer to a question calling for a proof is a proof, so let's get started. Since the integrand is positive throughout its interval of integration, we only need to show that the improper integral does not diverge to infinity. One proof involves the integral comparison test. For $x > 2$, $\log x < \sqrt{x}$ because $\log 2 < 2$ and because the derivative of $\log x$ is $1/x$ while the derivative of $\sqrt{x}$ is $(1/2\sqrt{x})$ which is larger.

   Now, by the integral comparison test, $\int_1^\infty f(x)\, dx < \int_1^4 f(x)\, dx + \int_4^\infty \sqrt{x}/x^2$. The first part of this upper bound is finite because the interval of integration is finite, and the second part is finite because it evaluates to 1. QED.

   Another proof involves integration by parts as well as comparison. First observe as in the previous proof that $1/x^2 > 1/(x^2 + 1)$. So we just need to prove that $\int_1^\infty x^{-2} \log x\, dx$ is finite. Now take

$U = \log x$, $dV = x^{-2}$, which yields $U = 1/x$ and $V = -x^{-1}$. Thus

$$\int_1^N \log x/x^2\,dx = -\left.\frac{\log x}{x}\right|_1^N + \int_1^N x^{-2}\,dx.$$

As N tends to infinity, the evaluation part tends to zero and the integral part tends to 1. So the overall upper bound integral is 1. The actual integral is, of course, less.

(b) Find a rational number A so that $A < L < A + 1/25$. In other words, find an approximation to L that's pins L down to within ± 0.04. Hint: For $x > 1$, $1/(x^2+1)$ can be written as a series of the form $p_1/x^2 + p_2/x^4 + p_3/x^6 + \cdots$.

My favorite answer here is $A = 8/9$. Getting this involves first observing that the series is actually $1/x^2 - 1/x^4 + 1/x^6 - 1/x^8 \cdots$. Thus

$$L = \sum_{n=0}^{\infty} (-1)^n \int_1^{\infty} \frac{\log x}{x^{2n+2}}\,dx.$$

These integrals evaluate in the same way as above, and the general answer is $\pm 1/(2n+1)^2$, so

$$L = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots.$$

This is an alternating series, so stopping right after a minus term gives us a lower bound, and stopping right after a plus sign gives us an upper bound. So $1 - 1/9 < L < 1 - 1/9 + 1/25$ and $A = 8/9$ will serve as an answer.

The actual number is known as Catalan's constant and often denoted G. Read all about it in the wikipedia article on *Catalan's constant*.

3. A thousand-meter deep borehole full of water has a tapered shape, with circular horizontal cross sections (whose centers are lined up vertically). The cross section of radius r is at depth $1000 - r^3$. (And r ranges from 0 to 10.)

   (a) Find the volume of the water in the borehole.

   The volume is 60000π cubic meters. This is the boring part. The volume is the integral of the depth. The depth at radius r is given to be $1000-r^3$. In polar coordinates, d area is given by $r\,dr\,d\theta$, or if everything is independent of θ, as here, one just writes $2\pi r\,dr$ and integrates only with respect to r. The volume (in cubic meters) is given by $\int_{r=0}^{10} 2\pi r(1000-r^3)\,dr = 60000\pi$.

(b) Find the work (in joules) required to lift all that water to the surface. ($G = 9.8$, water density$=$ 1000 kg per cubic meter.)

All the units are MKS units, so we can dispense with keeping track of the names for the units. The work is the integral of mass times distance stuff has to be lifted, times G. A vertical straw reaching from the surface down to a depth of $1000 - r^3$ meters will have volume $1000 - r^3$ times its surface area, and while some of the water is not as deep as other parts of it, the average depth of the water in the straw is $(1/2)(1000 - r^3)$. Thus we set up the problem as

$$W = 9.8 \times 1000 \int_{r=0}^{10} 2\pi r \frac{1}{2}(1000 - r^3)^2\,dr.$$

That comes to 220500000000π. Just for your amusement and mine, I converted this into megawatt hours. It came to a bit under 200 of those. It would be expensive to empty that hole, even if the pump was 100 percent efficient.

4. Find

$$\int_0^1 (x^4 + x^3 + 1)(6x^5 + 3x^2 + 1) + (4x^3 + 3x^2)(x^6 + x^3 + x)\, dx.$$

The answer is 9. The fun way to get the answer is to observe that the integrand has the form $fg' + f'g$, where $f = x^4 + x^3 + 1$ and $g = x^6 + x^3 + x$. That means that the integrand has the form $(fg)'$, so the integral itself is fg evaluated at 0 and 1. At 0, we get 0, and at 1, we get 9. So the answer is 9.

5. Find

$$\frac{d}{dx}\left[x \int_{t=1}^{\infty} e^{-x^2 t^2}\, dt\right].$$

The answer is e^{-x^2}. But why?

Doing the integral, and then taking the derivative of the answer, is a tall order. But integrals and derivatives are in some sense opposites, so it shouldn't matter. Except...the integral does not present in the form suitable for applying the fundamental theorem of calculus. But there is hope.

Make the substitution $u = xt$, $du = x\, dt$. With that substitution, the expression to be differentiated with respect to x becomes $\int_{u=x}^{\infty} e^{-u^2}\, du$. By the fundamental theorem of calculus, the derivative of this integral is $-e^{-x^2}$. And that's why.

6. Let

$$S = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

(a) Prove that S is finite.

The ratio test shows that it's finite. The limit of the ratio of numerators is 1, while the limit of the ratio of denominators is 2.

(b) Prove that $S < 8$. You may find the fact that for $n \geq 4$, $(n+1)^2/n^2 \leq 25/16$ helpful.

It is helpful. The first three terms are $1/2+4/4+9/8 = 21/8$. After that, the actual kth term $k^2/2^k$ is no more than $16 * (25/16)^(k-4)/2^k$, which is what it would be if the numerator increased by a ratio of 25/16 from then on, after k reached 4. But the sum of those terms is easy to get exactly because they're a geometric series, and better still, a geometric series in which the first term is 1. The common ratio is $25/32$, so the sum is $1/(1-25/32) = 32/7$. Adding $21/8 + 32/7 = 403/56 < 8$.

(c) Find the exact value of S.

It's 6. There are a couple of nice ways to see this. The first one is kind of slick, but hard to think of unless you've seen the trick before: consider the related sum $h(s) = \sum_{n=1}^{\infty} e^{ns}/2^n$. Introducing a parameter that isn't in the original problem is in the same spirit as introducing a line or circle or point that isn't part of the statement of a geometry problem: it's a good idea if the new thing is in some sense related to the question. Here, the point is that the derivatives of e^{ns} are ne^{ns} and n^2e^{ns}. So in particular

$$h''(s) = \sum_{n=1}^{\infty} n^2 e^{ns}/2^n.$$

Setting $s = 0$ gives the original problem, which means we can check off the requirement that the newly introduced thing be related to the question.

On the other hand, $h(s)$ is a geometric series and can be evaluated in closed form:

$$h(s) = \frac{e^s}{2 - e^s}$$

(for $s < \log 2$ only!, but that's OK because we are just interested in what happens for s near zero.) So

$$h'(s) = \frac{2e^s}{(2-e^s)^2}, \quad h''(s) = \frac{4e^s + 2e^{2s}}{(2-e^s)^3}.$$

Now, setting $s = 0$ gives 6.

The other approach is to observe that if we call the sum S, then

$$\begin{aligned}
S &= \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \cdots \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \\
&\qquad + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \cdots + \\
&\qquad\qquad + \frac{5}{8} + \frac{5}{16} + \frac{5}{32} + \cdots + \\
&\qquad\qquad\qquad + \ddots .
\end{aligned}$$

The first row of all this sums to 1, the second to 3/2, the third to 5/4, the next to 7/8, the next to 9/16, and so on. Another round of this approach finally yields the answer.