2017 Power Team
Texas A&M High School Students Contest
Solutions
October 2017

1. Two villages A and B are located on one side of a river with healing water. The river bank is a straight line. Alice lives in village A and her friend Bob lives in village B. One day Alice felt sick and she called Bob by phone asking him to bring her some fresh healing water from the river. So, Bob needs to go from village B to village A visiting the river. He decided to choose the shortest possible path to do this. Describe this path geometrically.

   Solution: (illustrated by Fig. 1)

   Assume that the river is represented by the line ℓ. Let A' be the point obtained from A by reflection with respect to the line ℓ.

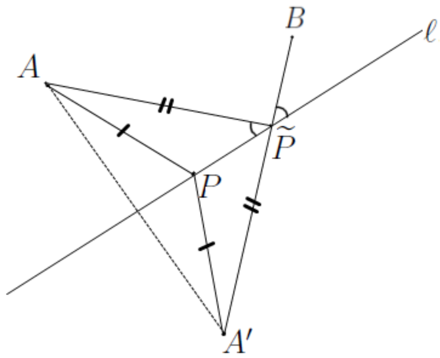

Fig.1

   From the fact that the shortest path between two points on a plane is the segment of the straight line connecting these points it follows that if the required shortest path meets ℓ at a point P, then this path must be along the broken line BPA. By the construction of A' we have $|AP| = |A'P|$, so the length of the broken line BPA is equal to the length of the broken line BPA' which is not smaller than $|BA'|$ and is equal to it if and only if this broken line is a segment of a straight line, i.e. when P is the point of intersection of $A'B$ with the line ℓ (this point is denoted by $\widetilde{P}$ on Fig.1).

   Hence the *required shortest path is along the broken line $B\widetilde{P}A$, where $\widetilde{P}$ is the point of intersection of $A'B$ with the line ℓ.*

   □

The following Lemma will be important in the sequel:

Lemma 1. *In the notations of the solution of Problem 1 the following properties hold:*

(a) The angle between $\widetilde{P}B$ and ℓ is equal to the angle between $\widetilde{P}A$ and ℓ, i.e. for the shortest path (and only for this path) the angle in which the path meets ℓ (the angle of incidence) is equal to the angle of departure from ℓ (the angle of reflection), i.e. as in the law of reflection in Geometric Optics or in billiards (see Fig. 2).

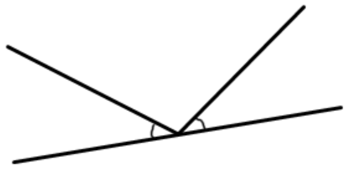

Fig.2

(b) If $P \neq \widetilde{P}$, then by shifting P toward $\widetilde{P}$ the length of the brocken line APB is decreased.

Proof. **1.** The first statement follows immediately from the construction in the solution of problem 1 (see Fig. 1):$\angle A\widetilde{P}P = \angle A_1\widetilde{P}P$ and the latter is equal to the angle between ℓ and $\widetilde{P}B$.

2. To prove the second statement we can assume that the line ℓ is in the direction of x-axis and the point A has coordinates $(0, h_1)$ (we can apply an appropriate rigid motion to achive this). Assume that point B has coordinates (a, h_2) and the point P has coordinates $(x, 0)$ (see Fig 3).

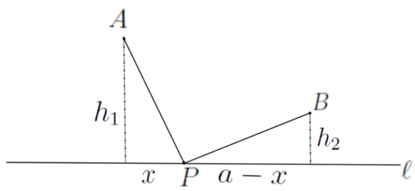

Fig. 3

Then the length of the broken line APB is described by the following function

$$f(x) = \sqrt{x^2 + h_1^2} + \sqrt{(a - x)^2 + h_2^2}.$$

Then

$$f'(x) = \frac{x}{\sqrt{x^2+h_1^2}} - \frac{a-x}{\sqrt{(a-x)^2+h_2^2}}$$

and $f'(x) = 0$ if and only if $\dfrac{x}{h_1} = \dfrac{a-x}{h_2}$. The latter equation has the unique solution $x = \tilde{x}$ that corresponds to the point $\widetilde{P}$ constructed in the solution of problem 1. By problem 1 $\tilde{x}$ is the global minimum of the function f and the fact that it is the unique critical point of f implies the statement of the lemma: the function f is strictly decreasing for $x < \tilde{x}$ and strictly increasing for $x > \tilde{x}$. $\square$

2. (a) An ant sits at a point M inside of a given acute angle in the plane. He needs to visit one side of the angle, then the other side of the angle, and then to return to M (note that visiting the vertex of the angle is considered as visiting both sides) . Describe geometrically the shortest path to perform this task.

   (b) Solve the same problem if the angle is obtuse or right.

Solution of item (a) (illustrated by Fig. 4) Denote the rays of the angle by ℓ_1 and ℓ_2 and the vertex of the angle by O. By analogy with the previous problem let M_1 and M_2 be the reflections of M with respect to ℓ_1 and ℓ_2, respectively. Since the angle is acute, the angle $\angle M_1OM_2$ is less than π (it is double of the original angle). Therefore the segment M_1M_2 intersects rays ℓ_1 and ℓ_2. Denote the points of intersection by $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively.

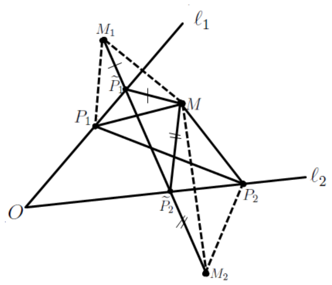

Fig. 4

Suppose that the ant visits ℓ_1 at point P_1 and visits ℓ_2 at point P_2, then to travel along the shortest path he must go along the triangle MP_1P_2. Since by constructions $MP_1 = M_1P_1$ and $MP_2 = M_2P_2$, the perimeter of the triangle MP_1P_2 is equal to the length of the broken line $M_1P_1P_2M_2$. The latter is not smaller than $|M_1M_2|$ and is equal to it if and only of $P_1 = \widetilde{P}_1$ and $P_2 = \widetilde{P}_2$.

Therefore, the *shortest path is along the triangle $M\widetilde{P}_1\widetilde{P}_2$, where $\widetilde{P}_1$ and $\widetilde{P}_2$ are the points of intersection of the segment M_1M_2 with the rays ℓ_1 and ℓ_2 respectively.*

Solution of item (b) If the angle is right then the segment M_1M_2 will pass through the vertex O of the angle (i.e $P_1 = P_2 = O$), so the shortest path will be first going along the segment MO and then back, which will be denote by MOM for shortness.

In the case of obtuse angle, in contrast to the previous cases, the segment M_1M_2 will not intersect the ray ℓ_1 and ℓ_2 so the length of every paths performing the task is strictly larger than $|M_1M_2|$. We claim that the shortest path in this case is the same as in the right angle case, i.e. MOM. Indeed, if the path meets ℓ_1 at P_1 and ℓ_2 at P_2 (the case when one or both of P_i is the vertex O is not excluded), then it is not shorter than the broken line MP_1P_2M. So, the shortest path can be found among those broken lines. Also , since the length of this line will go to infinity as $|OP_1|+|OP_2|$ goes to infinity, the shortest path exists by the extreme value theorem. Now it is impossible for the shortest path that both P_1 and P_2 are different from O, because in this case by Lemma 1 the path will satisfy the billiard property at both P_1 and P_2. i.e. for each ray ℓ_i the angles of incidence and reflection from it along the path are equal (otherwise, by Lemma 1 we can make the path shorter by slightly moving P_i). But this may happen only if the points M_1, P_1, P_2, and M_2 lie on the same line, which is impossible. So, one of the points P_i of the shortest path must be a vertex O but this implies that the shortest path is MOM.

3. (a) An ant crawls inside an acute triangle ABC. His task is to move along a closed path, visiting all sides of the triangle (if he visits a vertex it is considered as he visited both sides adjacent to this vertex). He can start anywhere inside the triangle. Describe the set of all points in which he can start to perform this task along the shortest possible path. Describe this path geometrically.

   (b) Solve the same problem if the triangle is not acute.

Solution of item (a): (illustrated by Fig. 5) Suppose that the ant meets the side BC at the point M (we do not exclude that M is one of the vertices here). Then we can repeat the constructions of the item (a) of the previous problem: The angle $\angle BAC$ is acute, the shortest path is along the triangle $M\widetilde{P}_1\widetilde{P}_2$, where $\widetilde{P}_1$ and $\widetilde{P}_2$ are the points of intersection of the segment M_1M_2 with the rays AB and AC and M_1, M_2 are the reflections of M with respect to the same rays, respectively. The length of this path is equal to $|M_1M_2|$.

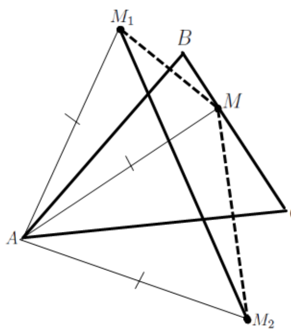

Fig. 5

Now we have to choose the point M on the side BC such the length of the corresponding segment M_1M_2 will be as short as possible. Note that the angle $\angle M_1AM_2$ is double of the $\angle BAC$ independently of the choice of the point M, because by constructions $\angle M_1AB = \angle MAB$ and $\angle M_2AC = \angle MAC$. Also, the triangle M_1AM_2 is isosceles, because

$$|AM_1| = |AM| = |AM_2| \tag{1}$$

Therefore $|M_1M_2|$ is minimal when $|AM_1|$ is minimal. By (1) it occurs when $|AM|$ is minimal among all M in BC, i.e. when AM is the altitude (here we also used the fact that the triangle ABC is acute). In other words, the shortest path is unique (up to the choice of the starting points) and it meets the side BC at the base of the altitude from A. Repeating the same arguments for other sides we get that the shortest path meets the side AB at the base of the altitude from C and it meets the side AC at the base of the altitude from B.

Conclusion The required shortest path is the triangle with the vertices in the bases of altitudes. This triangle is called *orthic triangle*. The ant must start at one of the points of the orthic triangle.

Solution of item (b): Suppose that $\angle A \geq \dfrac{\pi}{2}$ and the ants meets the side BC at the point M (we do not exclude that M is one of the vertices here). Then by the item (b) of the previous problem the shortest path (among the paths starting at M) is the broken line MAM and therefore *the shortest path of the problem is MAM, where AM the altitude from the vertex A with obtuse angle. The ant must start somewhere on this altitude.* $\square$

In problems 4-8 below an ant crawls inside a **convex quadrilateral** $ABCD$ and again his task is to move along a closed path, visiting all sides of the quadrilateral (if he visits a vertex it is considered as he visited both sides adjacent to this vertex). He can start anywhere inside the quadrilateral.

4. Assume that the quadrilateral $ABCD$ is a rectangle with sides of length a and b. Describe geometrically the shortest paths among all paths which perform the task and express the length of the shortest path in terms of a and b.

   Solution (illustrated by Fig. 7) Assume the ant visits the side AB at the point E, the side BC at the point F, the side CD at the point G and the side AD at the point H (some of these points might be equal and coincide with a vertex). Since the path is the shortest one it must be along the quadrilateral $EFGH$, because all other ways to connect the points E, F, G, and H into a closed path will give a longer path (see Fig. 7). Then the path of the ant is not shorter than the perimeter of the quadrilateral $EFGH$ (or , more precisely, than the length of the closed broken line $EFGHE$ if the quadrilateral degenerates to a triangle or a segment), because all other ways to connect the points E, F, G, and H into a closed broken line will give a longer path (see Fig. 6).

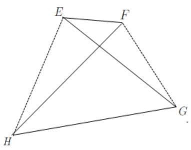

Fig. 6

   Now let E_1 be the reflection of E with respect to the line BC, E_2 is the reflection of E with respect the line AD, and finally E_3 is the reflection of E_2 with respect to the line CD (see Fig. 6). Let H_1 be the reflection of H with respect to the line CD. By constructions, using the fact that a reflection with respect to the line preserves the distances between points, $|EF| = |E_1F|$, $|GH| = |GH_1|$, and $|HE| = |HE_2| = |H_1E_3|$. Therefore the perimeter of the quadrilateral $EFGH$ is equal to the length of the broken line $E_1FGH_1E_3$, i.e it is not shorter than $|E_1E_3|$.

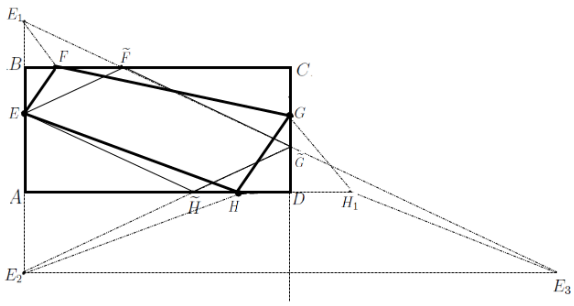

Fig. 7

   On the other hand the segment E_1E_3 intersects the sides BC and CD. Denote the points of intersection by $\widetilde{F}$ and $\widetilde{G}$, respectively, and let $\widetilde{H}$ be the point of intersection of E_2G with the side AD. Then the perimeter of the quadrilateral $E\widetilde{F}\widetilde{G}\widetilde{H}$ is equal to $|E_1E_3|$, so that the quadrilateral $E\widetilde{F}\widetilde{G}\widetilde{H}$ is the shortest possible path. Note that this is **an inscribed parallelogram with sides parallel to the diagonals of the rectangle**. Also if $E = B$ then this path degenerates to the broken line BDB (i.e. the motion is along the diagonal BD and back) and if $E = A$, then this path degenerates to the broken line ACA (i.e. the motion is along the diagonal AC and back). The length of the shortest paths is equal to $|E_1E_3| = 2|BD| = \boxed{2\sqrt{a^2+b^2}}$.

5. Assume that the quadrilateral $ABCD$ satisfies the following condition: among all paths that perform the task there is a path of shortest possible length which does not meet a vertex of the quadrilateral. Prove that the quadrilateral $ABCD$ can be inscribed into a circle.

   Proof. In the sequel by a quadrilateral inscribed in the quadrilateral $ABCD$ we mean a convex quadrilateral with all vertices lying on different sides of $ABCD$.

   Definition 1. *We say that a quadrilateral inscribed into the quadrilateral $ABCD$ satisfies the* **billiard property** *with respect to $ABCD$ if it makes equal angles with each side of the given quadrilateral $ABCD$*

   Now as a consequence of Lemma 1 we have the following

   Lemma 2. *The shortest inscribed quadrilateral $EFGH$ must satisfy the billiard property with respect to $ABCD$.*

   Proof. Assume by contradiction that the sides of $EFGH$ adjacent to one of the vertex, for example, the vertex E do not make equal angles with AB, then from Lemma (1) by an arbitrary small shift E along the side AB (toward the point that solves Problem with points F and H and the line ℓ connecting A and B) decreases the length of the inscribed quadrilateral, contradicting the assumption on shortness of $EFGH$. □

   So we can assume (see Fig. 8) that

$$\angle AEH = \angle BEF = \alpha, \angle BFE = \angle CFG = \beta$$
$$\angle CGF = \angle DGH = \gamma, \angle DHG = \angle AHE = \delta$$

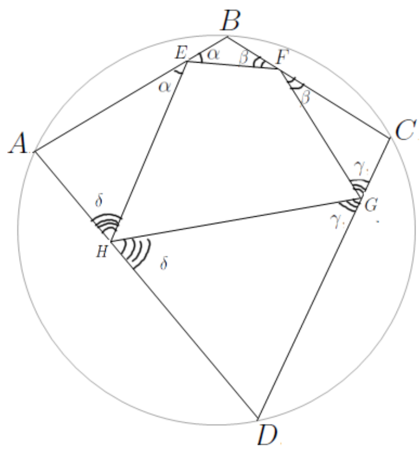

Fig. 8

Then $\angle HEF = \pi - 2\alpha$, $\angle EFG = \pi - 2\beta$, $\angle FGH = \pi - 2\gamma$, and $\angle GHE = \pi - 2\delta$, which together with the fact that the sum of all angles in a quadrilateral is equal to 2π implies that

$$\alpha + \beta + \gamma + \delta = \pi. \tag{2}$$

On the other hand,

$$\angle BAD = \pi - \alpha - \delta, \quad \angle BCD = \pi - \beta - \gamma,$$

which together with (2) implies that

$$\angle BAD + \angle BCD = \pi,$$

i.e. the sum of the opposite angles in the quadrilateral $ABCD$ *is equal to* π, which is equivalent to the fact that the quadrilateral $ABCD$ can be inscribed into a circle. Such quadrilaterals are also called *cyclic.* □

6. Assume that the quadrilateral $ABCD$ satisfies the condition of the previous problem, i.e., among all paths that perform the task, there exists a path of the shortest possible length which does not meet a vertex of the quadrilateral. Show that among all paths that perform the task there are infinitely many different shortest paths (note that here we do not distinguish paths that trace the same closed curve with different starting points and/or directions of motion). Describe how to construct these shortest paths.

Solution

We start with the proof of several Lemmas.

Lemma 3. *Fix a point E on the side AB. Let E_1 be the reflection of E with respect to the line BC, E_2 is the reflection of E with respect the line AD, and finally E_3 is the reflection of E_2 with respect to the line CD (see Fig. 7). Then the length of any path performing the task and starting from E is not smaller than $|E_1E_3|$.*

Proof. (illustrated by Fig. 9) First, for simplicity assume that a path Γ starting at E does not meet a vertex of $ABCD$. Then it must contain at least one point inside each side. Assume that Γ meets the side BC at a point F , the side CD at a point G , and the side AD at a point H . The length of Γ is not smaller than the perimeter of the quadrilateral $EFGH$. Let H_1 be the reflection of H with respect to the line CD. By construction, using the fact that a reflection with respect to the line preserves the distances between points, $|EF| = |E_1F|$, $|GH| = |GH_1|$, and $|HE| = |HE_2| = |H_1E_3|$.

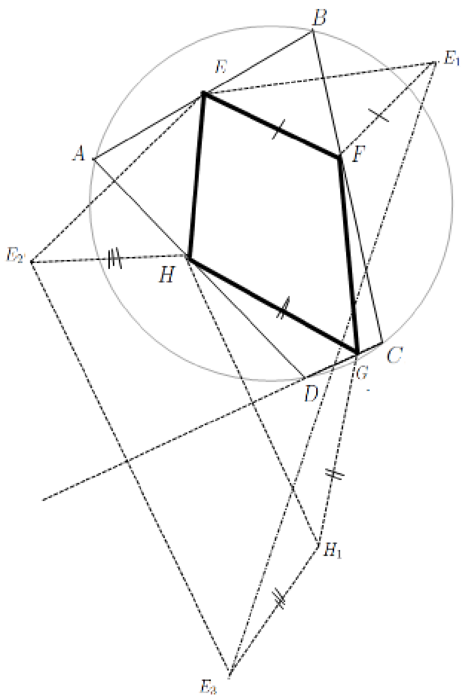

Fig. 9

Therefore the perimeter of the quadrilateral $EFGH$ is equal to the length of the broken line $E_1FGH_1E_3$, i.e it is not shorter than $|E_1E_3|$. The case when Γ meets a vertex of $ABCD$ can be treated similarly: a path Γ can be replaced without increasing the length by either a triangle with one vertex in a vertex of $ABCD$ and two other vertices inside the sides which are not adjacent to the first vertex or by a segment between two opposite vertices of $ABCD$ elapsed twice. In both cases by the same arguments the length of of the resulting path is not smaller than E_1E_3 (just some lags of the corresponding broken line will degenerate to points).

$\square$

Lemma 4. *Let Γ be a path which starts at E inside the side AB, does not meet a vertex of the quadrilateral $ABCD$ and it is the shortest path among all paths that perform the task and start at E. Then the lower bound $|E_1E_3|$ of the previous Lemma is achieved on the Γ (in other words, the length of path Γ is equal to $|E_1E_3|$).*

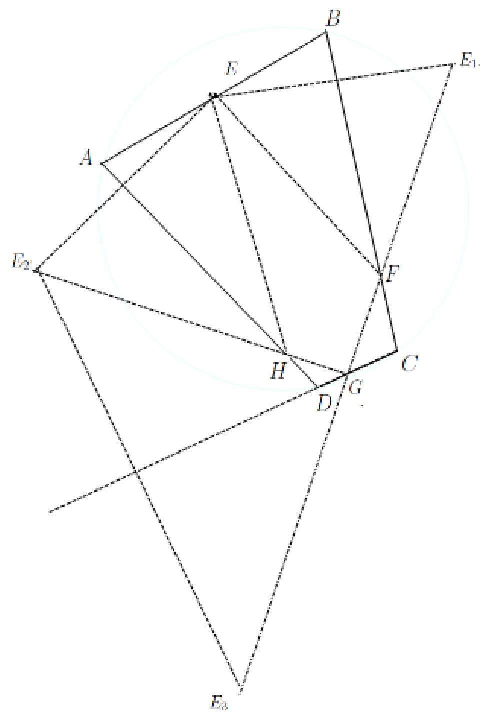

Fig. 10

Proof. (Illustrated by Fig. 10) By the same arguments as in the beginning of the proofs of the previous problem and the previous lemma Γ must be the quadrilateral $EFGH$, where F lies inside the side BC, G lies inside the side CD and H lies inside the side AD (recall that E lies inside the side AB by the assumptions). Let H_1 be the reflection of H with respect to the line CD. Then from Lemma 2 it follows that the points F, G, and H_1 lie on the segment E_1E_3, i.e. the perimeter of the quadrilateral $EFGH$ is equal to $|E_1E_3|$.

□

The last two lemmas give an explicit way to construct the shortest path which starts at the point E inside the side AB :

Corollary 1. *If there exists the shortest path as in the previous lemma, starting at the point E inside the side AB then this path is the quadrilateral $EFGH$ such that F is the intersection of the segment E_1E_3 with the side BC, G is the intersection of the segment E_1E_3 with the side CD, and H is the intersection of the segment E_2G with the side AD.*

Now take another point E' on the side AB and construct from it the points E'_1, E'_2, E'_3 by the same series of reflections as we did for the point E in Lemma 3.

Lemma 5. *(see Fig. 11) If the quadrilateral $ABCD$ is cyclic, then the vectors $\overrightarrow{E_1E_3}$ and $\overrightarrow{E'_1E'_3}$ are equal. In particular, $|E_1E_3| = |E'_1E'_3|$ and the lines E_1E_3 and $E'_1E'_3$ are parallel.*

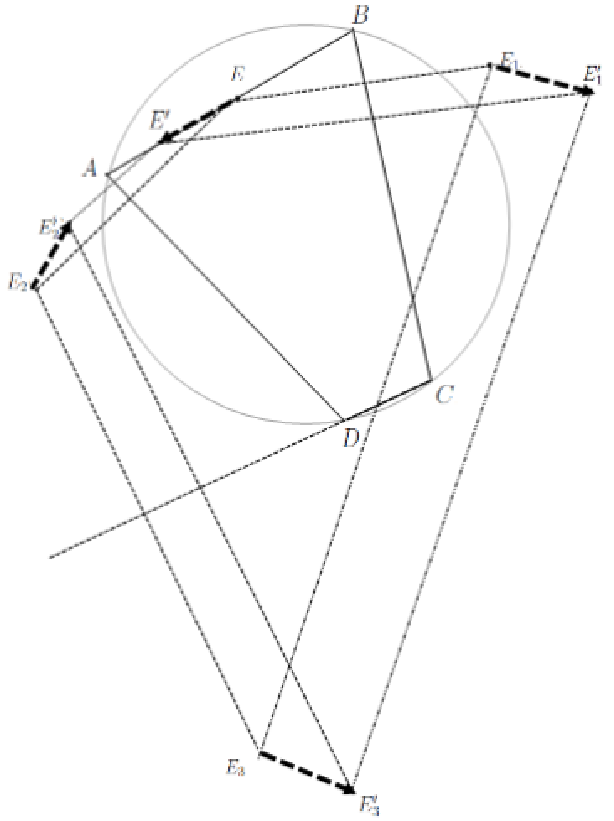

Fig. 11

Proof. Since E_1' and E_3' are obtained from E_1 and E_3 by a composition of three reflections and reflections preserve the distance between points, we have that $|E_3E_3'| = |E_1E_1'|$.

It remains to prove that vectors $\overrightarrow{E_1E_3}$ and $\overrightarrow{E_1'E_3'}$ are in the same direction, i.e. that the angle between these vectors is equal to 0. In general, by an angle $\angle(\vec{a}, \vec{b})$ between two vectors $\vec{a}$ and $\vec{b}$ we mean the signed angle (defined modulo 2π) with positive sign corresponding to the counter-clockwise rotation from $\vec{a}$ to $\vec{b}$. Note that if a vector $\vec{a}$ has an angle θ with a vector $\vec{b}$, then the angle between $\vec{a}$ and its reflection with respect to the line generated by b is equal to 2θ. Since, as was already mentioned, the vector $\overrightarrow{E_3E_3'}$ is obtained from the vector $\overrightarrow{E_1E_1'}$ by three reflections, let us find the angle between these vectors by studying the angles between consecutive reflections of $\overrightarrow{E_1E_1'}$.

The angle at the vertex X of the quadrilateral $ABCD$ will be denoted by $\angle X$. Without loss of generality we can assume that E is closer to B than E' (in this proof the points E and E' have equal role). Then

- $\angle\big(\overrightarrow{E_1E_1'}, \overrightarrow{CB}\big) = \pi - \angle B$. Therefore $\angle\big(\overrightarrow{E_1E_1'}, \overrightarrow{EE'}\big) = 2\pi - 2\angle B \equiv -2\angle B$;
- $\angle\big(\overrightarrow{EE'}, \overrightarrow{DA}\big) = -\angle A$. Therefore $\angle\big(\overrightarrow{EE'}, \overrightarrow{E_2E_2'}\big) = -2\angle A$;
- Let A_1 be the orthogonal projection of A to the line CD. Since $ABCD$ is a cyclic quadrilateral, $\angle D = \pi - \angle B$. Then

$$\angle A_1AD = \left|\frac{\pi}{2} - \angle D\right| = \left|\angle B - \frac{\pi}{2}\right|$$

(in Fig.11 the case of acute $\angle B$ is shown). Further, treating the cases of acute and obtuse $\angle B$ separately, one gets the following uniform formulas $\angle\big(\overrightarrow{E_2'E_2}, \overrightarrow{A_1A}\big) = \angle A + \angle B - \frac{\pi}{2}$ and

$$\angle\big(\overrightarrow{E_2E_2'}, \overrightarrow{CD}\big) = -\Big(\frac{\pi}{2} - \angle\big(\overrightarrow{E_2'E_2}, \overrightarrow{A_1A}\big)\Big) = \angle A + \angle B - \pi.$$

Consequently,

$$\angle\big(\overrightarrow{E_2E_2'}, \overrightarrow{E_3E_3'}\big) \equiv 2\angle A + 2\angle B.$$

Combining all three items together we get that

$$\angle\big(\overrightarrow{E_1E_1'}\overrightarrow{E_3E_2'}\big) \equiv -2\angle A - 2\angle B + 2\angle A + 2\angle B = 0,$$

i.e. $\overrightarrow{E_1E_1'}$ and $\overrightarrow{E_3E_3'}$ are in the same direction. $\square$

As a consequence of the previous lemma we have the followng

Corollary 2. *For a cyclic quadrilateral $ABCD$ the lower bound given by Lemma 3 for the length of the paths performing the task and starting at a point E of the side AB is in fact independent of this point, i.e. it is the lower bound for all paths performing the task.*

Note that in general not for every point E on the side AB we can implement the construction of Corollary 1: for example, E_1E_3 may not even intersect the sides BC (or the constructions of Corollary 1 may fail in every other step). However, from the assumpion of the problem and Lemma 4 it follows that there exists a point E inside AB for which the constructions of Corollary 1 can be implemented. Then it can be implemented for any point E' sufficiently close to E, i.e. for E' sufficiently closed to E the segment $E_1'E_3'$ intersects the side BC at a point F' and the side CD at a point G', while the segment $E_2'G'$ intersects the side AD at a point H' and the quadrilateral $E'F'G'H'$ has length $|E_1'E_3'| = |E_1E_3|$ by Lemma 5, i.e. this is also the shortest path different from $EFGH$. This shows that we have infinitely many different shortest paths.

Note that by construction the corresponding sides of the quadrilaterals $E'F'G'H'$ and $EFGH$ are parallel, i.e. $E'F' \parallel EF$, $F'G' \parallel F'G'$, $G'H' \parallel GH$, and $H'E' \parallel HE$. Therefore (see Fig. 12) in order to construct a new shortest path $E'F'G'H'$ from the known shortest path $EFGH$ (where E and E' inside AB) we can proceed as follows: First draw through E' the line parallel to EF. If the point F' of intersection of this line with the line BC does not lie inside the side BC we stop. If it lies inside the side BC, then draw through F' the line parallel to FG. If the point G' of intersection of this line with the line CD does not lie inside the side CD we stop. If it lies inside the side CD, then draw through G' the line parallel to GH. If the point H' of intersection of this line with the line AD does not lie inside the side AD we stop. If it lies we are done with the construction of the required new shortest path $E'F'G'H'$. For E' sufficiently closed to E we can implement the construction to the end without stopping. Finally note that if some vertices in the constructed family of shortest quadrilaterals approach to a vertex of the quadrilateral $ABCD$ then the shortest inscribed quadrilaterals can degenerate to a triangles and even to a segment between two opposite vertices, when we move along the segment twice in opposite directions, as in the case of the rectangle of Problem 4. With these degenerate cases we will get all possible shortest paths.

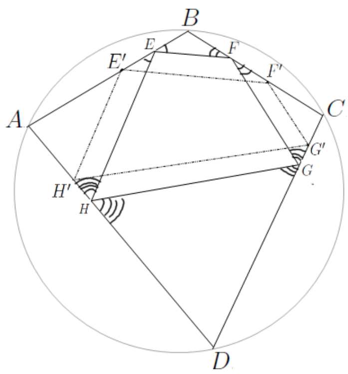

Fig. 12

7. Assume that the quadrilateral $ABCD$ can be inscribed into a circle and the vertices do not belong to any semicircle. For such a quadrilateral, prove that among all paths that perform the task there is a path of shortest possible length which does not meet a vertex of the quadrilateral.

   Proof. First we prove the following Lemma:

   Lemma 6. *If the quadrilateral $ABCD$ can be inscribed into a circle, then any quadrilateral which satisfies the billiard property with respect to $ABCD$ is the shortest path among all paths performing the task, i.e. in this case the converse of Lemma 2 holds true.*

   Proof. Assume that the quadrilateral $EFGH$ satisfies the billiard property with respect to $ABCD$ with vertex E lying inside the side AB. Then, as in the proof of Lemma 4, the perimeter of $EFGH$ is equal to $|E_1E_3|$ in the notation of Lemma 3, but the latter is the lower bound for all paths performing the task by Corollary 2. Therefore $EFGH$ is the shortest path. □

   So, we need to find at least one quadrilateral satisfying the billiard property with respect to $ABCD$. Let K be the point of intersection of the diagonals AC and BD. Let E , F, G, and H are the perpendicular projections of K onto the lines AB, BC, CD, and AD (see Fig. 13).

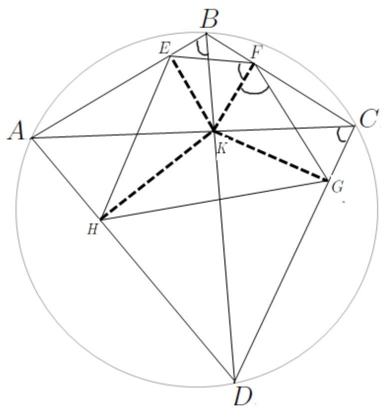

Fig.13

The condition that the vertices of $ABCD$ do not belong to any semicircle ensures that these projections lie inside the corresponding sides of $ABCD$, because in the triangles AKB, BKC, CKD, and AKD the angle of the vertices that are also vertices of the quadrilateral $ABCD$ are acute.

Let us prove that the the quadrilateral $EFGH$ satisfies the billiard property. Prove, for example, that $\angle EFB = \angle GFC$. In the quadrilateral $EBFK$ two angles are right, so it can be inscribed to a circle, therefore $\angle KFE = \angle EBF$ $(= \angle ABD)$ as supported by the same arc in this circle. In the same way, $\angle KFG = \angle KCG$ $(= \angle DCA)$. Note that $\angle ABD = \angle DCA$ as supported by the same arc in the cyclic quadrilateral $ABCD$. This proves that $\angle KFE = \angle KFG$, which implies that $\angle EFB = \angle GFC$. The billiard properties for other vertices of $EFGH$ are proved in the same way. □

8. Assume that the quadrilateral $ABCD$ cannot be inscribed into a circle. Describe the algorithm to find the shortest path among the paths performing the task (you can use the fact that the shortest path exists without justifying it).

   Solution By problem 5 the shortest path must meet a vertex of $ABCD$. Therefore we can proceed as follows: first construct the shortest path among all paths performing the task and starting from the given vertex and then compare the resulting paths obtained from the analysis of each of the four vertices.

   Let us describe how we find the shortest path among all admissible paths starting at vertex A. In this case the shortest path is a triangle AFG with E on the side CB and G on the side CD. We can proceed similarly to the Problem 2: the only difference is that in Problem 2 we do not require that the points E and F lie on the sides but on the rays from C, generated by the sides . So here, with additional restrictions, we have to consider more cases.

If the angle C is right or obtuse then by problem 2(a) the required shortest path is ACA.

If the angle C is acute take the reflection A_1 of A with respect to the line BC and the reflection A_2 of A with respect to the line CD (see Fig. 14).

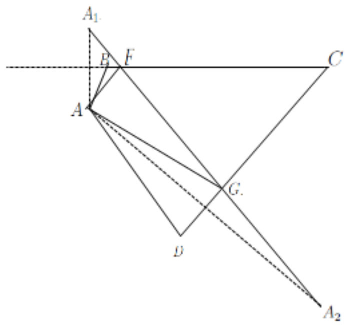

Fig. 14

Assume that F and G are the points of intersection of the line A_1A_2 with the rays CB and CD, respectively. Then

- If the points F and G lie inside the corresponding sides CB and CD of the quadrilateral $ABCD$ (as in Fig 14), then by problem 2(b) the required shortest path is the triangle AFG;
- If one of the points , say F, is outside the side CB and G is inside the side CD, (as in Fig 15), then by Lemma 1 the required shortest path is the triangle ABG;

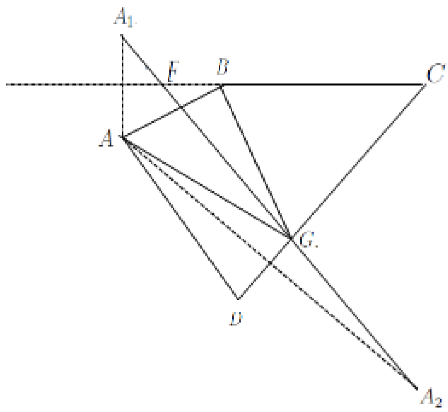

Fig.15

- If both points F and G are outside the corresponding sides of the quadrilateral $ABCD$, then again by Lemma 1 the required shortest path is the triangle ABD. However, this path is not the shortest path among all paths performing the task, because it is longer than the broken line BDB by the triangle inequality.

Repeat this algorithm for paths starting at each of other vertices B, C, and D, and compare the length of the resulting shortest paths to get the shortest path required in the problem.