Texas A&M University Algebra Qualifying Exam
Friday, August 14, 2015

Instructions: There are 8 problems, some of which have multiple parts. Please attempt them all.
Also:
- *Please start each problem on a new page, clearly writing the problem number and your name on that page.*
- Electronic or written assistance of any kind (such as calculators, cell phones, computers, notes, or books) is forbidden.
- Please do not interpret any problem in a way that renders it trivial.
- Unless otherwise specified, we assume rings to be commutative, with multiplicative identity. For any ring R we let R^* denote the group of units of R, we let $R^{n\times n}$ denote the set of $n \times n$ matrices with entries in R, and we let $\mathbb{F}_q$ denote the (finite) field with q elements. Also, we assume all R-modules are unital left R-modules. Finally, for any subset $S \subseteq R$ we use $\langle S\rangle$ to denote the ideal generated by S.

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1. Please prove that, up to isomorphism, there are at most 4 groups of order 306 containing an element of order 9.

2. Suppose $A \in \mathbb{Z}^{n\times n}$ has (i,j)-entry $a_{i,j}$ for all i,j, and $x=(x_1,\ldots,x_n)$. Let us then define x^A to be the vector of (formal) monomials $\left(x_1^{a_{1,1}}\cdots x_n^{a_{n,1}},\ldots,x_1^{a_{1,n}}\cdots x_n^{a_{n,n}}\right)$. One can prove (and you may assume) that $x^{AB}=(x^A)^B$ for any $B\in\mathbb{Z}^{n\times n}$.

(a) Please prove that when $\det(A)\in\{-1,1\}$, and K is a field, the function $m_A(x):=x^A$ defines an automorphism of $(K^*)^n$.

(b) For arbitrary $A\in\mathbb{Z}^{n\times n}$ the function $m_A(x):=x^A$ happens to define an endomorphism of the subgroup $\{-1,1\}^n$ of $(\mathbb{Q}^*)^n$. Please find, and prove, an explicit formula for the cardinality of the quotient group $\{-1,1\}^n/\mathrm{Ker}(m_A)$ as a function of the $(\mathbb{Z}/2\mathbb{Z})$-rank of the mod 2 reduction of A.

3. Let K be a field.

(a) Please prove that, given any non-zero vector $v\in K^n$, v can be completed to a basis, i.e., we can find $v_2,\ldots,v_n\in K^n$ with $\{v,v_2,\ldots,v_n\}$ a basis for K^n.

(b) Under the additional assumption that K is algebraically closed, please prove that, given any $A\in K^{n\times n}$, there are matrices $U,V\in K^{n\times n}$, with V invertible and U upper-triangular, such that $A=V^{-1}UV$.

4. Given any ring R, and R-modules A,A',B,B',C,C', and R-module homomorphisms $f,f',g,g',\alpha,\beta,\gamma$ such that α and γ are monomorphisms and the diagram

$$\begin{array}{ccccccccc} \mathbf{O} & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & \mathbf{O} \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \mathbf{O} & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & \mathbf{O} \end{array} \qquad (1)$$

commutes and (1) has exact top and bottom rows, please prove that β is a monomorphism.

5. Suppose $p,q\in\mathbb{N}$, p is prime, q is a prime power, and $\mathbb{F}_q$ is a field with exactly q elements. Also let $\phi^{(k)}$ denote the k-fold composition of an endomorphism ϕ with itself.

(a) Please prove that if $x^{p^n}-x-1$ is irreducible in $\mathbb{F}_p[x]$ then (i) $\phi(y):=y^{p^n}$ defines an automorphism of $\mathbb{F}_p[x]/\langle x^{p^n}-x-1\rangle$, and (ii) $\phi^{(p)}$ is the identity map on $\mathbb{F}_p[x]/\langle x^{p^n}-x-1\rangle$.

(b) Suppose f is irreducible in $\mathbb{F}_q[x]$. Please prove that f divides $x^{q^n}-x$ if and only if the degree of f divides n.

(c) Please prove that $x^{47^n}-x-1$ is *not* irreducible in $\mathbb{F}_{47}[x]$ for $n\geq 2$.

6. Suppose $p\geq 3$ is prime.

(a) Please prove that $F(x):=x^p$ defines an automorphism of $\mathbb{F}_{p^2}$ fixing $\mathbb{F}_p$.

(b) Please prove that the polynomial x^p+x-2 has exactly p roots in $\mathbb{F}_{p^2}$.

7. Suppose M is a faithful R-module with the property that, if $m_1,m_2\in M$, then either $Rm_1=Rm_2$ or $Rm_1\cap Rm_2=\{0\}$. Please prove that either (i) M is irreducible or (ii) R is a field.

8. Please find an explicit univariate polynomial in $\mathbb{Z}[x]$ with Galois group $\mathbb{Z}/210\mathbb{Z}$ over $\mathbb{Q}$, and prove why your polynomial satisfies this property.