Algebra Qualifying Examination
11 August 2021

Instructions:

- There are nine questions worth a total of 100 points.
- **Read all problems first**. Make sure that you understand each and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit is awarded based both on the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and be legible. State clearly any major theorems you use (hypotheses and conclusions). Justify your reasoning.
- Start each problem on a new page, clearly marking the problem number and your name on that page. Do 'scratch work' on a separate page.
- Rings always have an identity (otherwise they are rng) and all R-modules are left modules.

---

(1) [13pts] Let V be a vector space of finite dimension n over a field F, and $T: V \to V$ an F-linear map.

 (a) Assuming that F is algebraically closed, prove that V has at least $n+1$ T-invariant subspaces. (A subspace $W \subset V$ is *T-invariant* if $T(W) \subset W$.)

 (b) Give an example showing the conclusion to (a) may be false when F is not algebraically closed.

 (c) Suppose that T is diagonalizable over F and has n distinct eigenvalues. How many T-invariant subspaces does V have?

(2) [12pts] Let G be a finite group of order greater than 2. Show that $|\operatorname{Aut}(G)| > 1$. (Hint: there are two cases: G is abelian and G is not abelian.)

(3) [5pts] Define free abelian groups in terms of a universal property. Specifically, what does it mean for an abelian group F to be free on a set $B \subset F$?

(4) [10] Let G be a group of order p^n, where p is a prime number and $n \geq 0$ an integer. Prove that for every integer p^t dividing p^n, G has a subgroup of order p^t. Hint: first show that when $n > 0$, G has a nontrivial center.

(5) [10pts] Prove that in a principal ideal domain every irreducible element is prime.

(6) [10pts] In each part of this problem determine if the given objects are isomorphic.

(a) $\mathbb{R}$ and $\mathbb{C}$ as $\mathbb{Q}$-vector spaces.
(b) $\mathbb{C}$ and $\mathbb{R} \times \mathbb{R}$ as rings.
(c) $\mathbb{R}[x]/\langle x^2 - 1 \rangle$ and $\mathbb{R} \times \mathbb{R}$ as rings.

(7) [12pts] Let R be an integral domain. Let I and J be ideals of R such that $I + J = R$.

(a) Show that $I \oplus J \cong (I \cap J) \oplus R$ as left R-modules. (Hint: Consider the natural map $I \oplus J \to R$.)
(b) If $I \cap J$ is a principal ideal, show that both I and J are projective R-modules.

(8) [12pts] Suppose that the field E is a Galois extension of the field F of order $2021 = 43 \cdot 47$. Prove that there exist intermediate fields K, L between F and E that are Galois extensions of F, that generate E ($KL = E$) and are disjoint over F, ($K \cap L = F$). (You may want to first understand $\mathrm{Gal}(E/F)$.)

(9) [16pts] Let $\zeta = e^{2\pi i/5}$ be a fifth root of unity, and let $K = \mathbb{Q}(\zeta)$.

(a) Prove that $K/\mathbb{Q}$ is a Galois extension.
(b) Let $G = \mathrm{Gal}(K/\mathbb{Q})$. For each $\sigma \in G$, show that $\sigma(\zeta) = \zeta^{a(\sigma)}$ for some $a(\sigma) \in (\mathbb{Z}/5\mathbb{Z})^\times$. Prove that the function $a\colon G \to (\mathbb{Z}/5\mathbb{Z})^\times$ is a group isomorphism.
(c) Let $L = \mathbb{Q}(\zeta, \sqrt[5]{3})$. Show that $L/\mathbb{Q}$ is a Galois extension and that there is a group isomorphism

$$\varphi : \ \mathrm{Gal}(L/\mathbb{Q}) \ \xrightarrow{\sim} \ \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \ \middle| \ a \in (\mathbb{Z}/5\mathbb{Z})^\times , \ b \in \mathbb{Z}/5\mathbb{Z} \right\} \subseteq \ \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z}) \, .$$

(Hint: For $\tau \in \mathrm{Gal}(L/\mathbb{Q})$, what is $\tau(\zeta)$? What can $\tau(\sqrt[5]{3})$ be?)