Algebra Qualifying Examination
6 January 2012

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations, but it may not be used as a step in your reasoning.
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.

Notation: Throughout, $\mathbb{Q}$ and $\mathbb{C}$ denote the field of rational or complex numbers, respectively. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{F}_p$ denotes the field with p elements, where p is a prime number.

1. (11 points) Let K be a field which is an extension of degree n of another field F, i.e. $F \subseteq K$ and $[K : F] = n$. Prove that K is isomorphic to a subring of the ring of $n \times n$ matrices over F. (Thus the ring of $n \times n$ matrices over F contains an isomorphic copy of every extension of F of degree $\leq n$.)

2. (12 points) Let R be a commutative ring with 1_R, and let $X = \{f_i\colon\ i \in\ I\}$ be a subset of R such that the ideal generated by X is the unit ideal $\langle 1_R \rangle = R$.

   a) Show that a *finite* number of elements of X generate the unit ideal.

   b) Say $f_1, \ldots, f_k \in X$ generate the unit ideal. Show that $f_1^{n_1}, \ldots, f_k^{n_k}$ generate the unit ideal for $n_1, \ldots, n_k$ fixed positive integers.

   c) Denote by R_{f_ℓ} the localization of R at the multiplicative set $S_\ell = \{1_R, f_\ell, f_\ell^2, f_\ell^3, \ldots\}$, i.e. $R_\ell = S_\ell^{-1}R$, and let $\varphi_\ell\colon\ R \to R_\ell$ be the canonical homomorphism. Consider two elements a and a' of R and suppose $\varphi_\ell(a) = \varphi_\ell(a')$ for $1 \leq \ell \leq k$. Show $a = a'$. (We are assuming as in b) that $f_1, \ldots, f_k$ generate the unit ideal.)

3. (14 points)

   a) Let G be a simple group of order n with a proper subgroup H of index $k > 1$, i.e. $[G : H] = k > 1$. Show that G is isomorphic to a subgroup of S_k, the symmetric group on k letters.
   b) Suppose P is a Sylow p-subgroup of G where we view G as a subgroup of S_k as above. Show that if P is also a Sylow p-subgroup of S_k then the order of the normalizer of P in G, i.e. $|N_G(P)|$, divides $|N_{S_k}(P)|$.
   c) Use 3a and 3b above to show that a group of order 396 cannot be simple. (Hint: Let $p = 11$ and $H = N_G(P)$.)

4. (14 points)

   a) Recall that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain and hence both a PID and a UFD. Use Eisenstein's criteria to show that $X^4 - 3$ is irreducible over the field $\mathbb{Q}(i)$.
   b) What is the Galois group of the splitting field F of $X^4 - 3$ over $\mathbb{Q}(i)$ as an extension of $\mathbb{Q}$, i.e. given $\mathbb{Q} \subset \mathbb{Q}(i) \subset F$ find $\mathrm{Gal}(F/\mathbb{Q})$. Justify your answer.
   c) Determine all the intermediate fields K with $\mathbb{Q} \subset K \subset F$.

5. (10 points) Let R be a ring with identity 1_R (not necessarily commutative). Show that the following conditions on a unitary (left) R module M are equivalent.

   i) M is injective
   ii) every short exact sequence $0 \to M \to B \to C \to 0$ (of unitary left R-modules) is split exact, i.e. $B \cong M \oplus C$.

6. (13 points)

   a) Consider the symmetric group S_n with $n \geq 3$. Suppose N is a normal subgroup of S_n that contains a 3-cycle. Show N contains every 3-cycle.
   b) Show that $N = A_n$ or S_n.

7. (13 points) Let V be a two dimensional vector space over the field $\mathbb{F}_p$ with p elements (p an odd prime). Let $L\colon\ V \to V$ be a linear transformation on V such that $L^{p-1} = I$, the identity map. Prove that L is diagonalizable, that is prove that there is a basis $\mathcal{B}$ for V such that $[L]_{\mathcal{B}}$, the matrix of L with respect to $\mathcal{B}$, is a diagonal matrix.

8. (13 points) Let R be a commutative ring with 1_R, and let $I \subsetneq R$ be a proper ideal. Show that there exists a minimal prime ideal P over I, i.e. a prime ideal P such that $I \subseteq P$ and such that there does not exist another prime ideal P' with $I \subseteq P' \subsetneq P$.