TEXAS A&M UNIVERSITY
ALGEBRA QUALIFYING EXAM
JANUARY 2015

INSTRUCTIONS:.

- There are 8 problems. Work on all of them.
- Prove your assertions.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Problem 1.

(a) Let G be a group and A and B abelian subgroups of G. Prove that $A \cap B$ is a normal subgroup of $\langle A \cup B \rangle$.
(b) Let G be a finite group which is not cyclic of prime order and in which every proper subgroup is abelian. Prove that G contains a nontrivial, proper, normal subgroup.

Problem 2.
Let G be a group of order 45. Prove that G is abelian.

Problem 3.

Let R be an integral domain which is noetherian (every ideal is finitely generated). Prove that, if every pair of nonzero elements $a, b \in R$ has a common divisor that can be written as an R-linear combination $xa+yb$ of a and b, for some $x, y \in R$, then R is a principal ideal domain.

Problem 4.

Prove that the polynomial $x^4 + x^2 + x + 1$ is irreducible over $\mathbb{Q}$.

Problem 5.

Consider the polynomial $f = x^5 - 6x + 3$ over $\mathbb{Q}$ and its splitting field F.

(a) Prove that f is irreducible over $\mathbb{Q}$.
(b) Prove that the Galois group G of the extension F over $\mathbb{Q}$ is a subgroup of S_5.
(c) Prove that G contains a 5-cycle.
(d) Prove that G contains a transposition.
(e) Determine G.

Hint 1: If you do not know how to do some part of the problem, skip it and assume it in the next part of the problem.

Hint 2: In part (d) take for granted that f has exactly 3 real roots.

Problem 6.

Prove that $\mathbb{Q}(\sqrt[4]{2})$ is not the splitting field of any polynomial over $\mathbb{Q}$.

Problem 7.

Let A, B and C be left modules over the commutative ring R (with identity) and let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence (in other words i is injective R-module homomorphism, p is surjective R-module homomorphism, and $Ker(p) = Im(i)$). Prove that there exists an R-module homomorphism $j : C \to B$ such that $pj = 1_C$ if and only if there exists an R-module homomorphism $q : B \to A$ such that $qi = 1_A$.

Problem 8.

Let R be a commutative ring with identity, I a prime ideal of R, and S the complement of I in R. Prove that the quotient ring $S^{-1}R$ is local.