Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 11, 2016

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = \phi''(x)$ for some $\phi \in \mathcal{D}$ if and only if $\int_{-\infty}^{\infty} \psi(x)dx = \int_{-\infty}^{\infty} x\psi(x)dx = 0$.
(d) Use 2(c) to solve, in the distributional sense, the differential equation $u'' = 0$.

Problem 2. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having u'' in $L^2[0,\infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0,\infty) \,|\, u'' \in L^2[0,\infty) \text{ and } u'(0) = 0\}.$$

(a) Find the Green's function $G(x,\xi;z)$ for $-G'' - zG = \delta(x-\xi)$, with $G_x(0,\xi;z) = 0$.
(b) Employ the spectral theorem (Stone's formula) to obtain the cosine transform formulas:

$$F(\mu) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\mu x)dx \text{ and } f(x) = \int_0^\infty F(\mu)\cos(\mu x)d\mu\,.$$

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) Consider $K \in \mathcal{C}(\mathcal{H})$. Show that if $\{\phi_n\}_{n=0}^\infty$ is an orthonormal set in $\mathcal{H}$, then $\lim_{n\to\infty} K\phi_n = 0$.
(b) Suppose that $K \in \mathcal{C}(\mathcal{H})$ is self adjoint.
   (i) Show that $\sigma(K)$ (the spectrum) consists only of eigenvalues, together with 0, and that the only limit point of $\sigma(K)$ is 0.
   (ii) Given that $\|K\| = \sup_{\|u\|=1} |\langle Ku, u\rangle|$, show that either $\|K\|$ or $-\|K\|$ (or possibly both) is an eigenvalue of K, and that the corresponding eigenspace is finite dimensional.

Problem 4. Suppose that $f(x)$ is 2π-periodic function in $C^{(m)}(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2π-periodic. Here $m > 0$ is a fixed integer. Let c_k denote the k^{th} (complex) Fourier coefficient for f, and let $c_k^{(j)}$ denote the k^{th} (complex) Fourier coefficient for $f^{(j)}$.

(a) Prove that $c_k^{(j)} = (ik)^j c_k$, $j = 0, \ldots, m+1$. (Note: using term by term differentiation of the Fourier series *assumes* what you want to prove.)
(b) For $k \neq 0$, show that c_k satisfies the bound

$$|c_k| \le \frac{1}{2\pi|k|^{m+1}}\|f^{(m+1)}\|_{L^1[0,2\pi]}.$$

(c) Let $f_n(x) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the n^{th} partial sum of the Fourier series for f, $n \ge 1$. Show that

$$\|f - f_n\|_{L^2[0,2\pi]} \le C\frac{\|f^{(m+1)}\|_{L^1[0,2\pi]}}{n^{m+\frac{1}{2}}},$$

where C is independent of f and n.

APPLIED MATHEMATICS/NUMERICAL ANALYSIS QUALIFIER

Aug. 11, 2016

Numerical Analysis part, 2hours

Problem 1. Let $K \subset \mathbb{R}^2$ be a simplex. Let $k \in \mathbb{N}$ and consider the set of multi-indices $\mathcal{A}_{k,2} := \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid |\alpha| \le k\}$. Let $\mathbb{P}_{k,2}$ be the set of the real-valued 2-variate polynomials of degree at most k.

(1) Let $\Sigma_{k,2}$ be the collection of the following linear forms $\sigma_\alpha(p) = \int_K \partial^{\alpha_1}\partial^{\alpha_2} p \,\mathrm{d}x$, for all $\alpha \in \mathcal{A}_{k,2}$ and all $p \in \mathbb{P}_{k,2}$, where $\partial^{\alpha_i} p(x_1, x_2)$ is the α_i-th partial derivative of p with respect to x_i. Show that $(K, \mathbb{P}_{k,2}, \Sigma_{k,2})$ is a finite element. (*Hint:* induction on k.)

(2) Let $f \in W^{k,1}(K)$. Show that there exists a unique $q \in \mathbb{P}_{k,2}$ such that $\int_K \partial^{\alpha_1}\partial^{\alpha_2}(q - f)\,\mathrm{d}x = 0$ for all $\alpha \in \mathcal{A}_{k,2}$.

(3) Compute the three shape functions of $(\widehat{K}, \mathbb{P}_{1,2}, \Sigma_{1,2})$ where $\widehat{K} := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1,\ 0 \le x_2,\ x_1 + x_2 \le 1\}$.

Problem 2. Consider the equation $\mu\partial_t u + \beta\partial_x u - \nu\partial_{xx} u = f$ in $D = (0, 1)$, $t > 0$, where $\mu \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, $\nu \in \mathbb{R}_+$ and $f \in L^2(D)$ with boundary conditions $u(0) = 0$, $u(1) = 0$ and initial data $u(x, 0) = 0$. Let $\mathcal{T}_h$ be the mesh composed of the cells $[ih, (i+1)h]$, $i \in \{0{:}I\}$, with uniform mesh size $h = \frac{1}{I+1}$. Let $P(\mathcal{T}_h)$ be the finite element space composed of continuous piecewise linear functions that are zero at 0 and at 1. Let $(\varphi_i)_{1\le i\le I}$ be the global Lagrange shape functions associated with the nodes $x_i := ih$, $i \in \{1{:}I\}$.

(1) Write the fully discretize version of the problem in $P(\mathcal{T}_h)$ using the implicit Euler approximation for the time. Denote the time step by Δt and $t^l := l\Delta t$ for all $l \in \mathbb{N}$.

(2) Prove one stability estimate. (*Hint:* You may want to introduce the Poincaré constant c_P such that $c_P\|v\|_{L^2} \le \|\partial_x v\|_{L^2}$ for all $v \in H^1_0(D)$.)

(3) Denoting by $u_h^l = \sum_{0\le i\le I+1} U_i^l \varphi_i$ the approximation of u at time $t^l = l\Delta t$, write the linear system solved by $(U_1^{n+1}, \ldots, U_I^{n+1})^\mathsf{T}$.

Problem 3. Let $D = (0, 1)$ and $f(x) = \frac{1}{x(1-x)}$. Consider the problem: $-\partial_x((1 + \sin(x)^2)\partial_x u) = f$ in D with $u(0) = u(1) = 0$.

(1) Prove that f is the weak derivative of $g(x) = \log(x) - \log(1-x)$. Is g in $L^2(D)$?

(2) Write a weak formulation of the above problem with both trial and test spaces equal to $H^1_0(D)$.

(3) Show that the problem is well posed in $H^1_0(D)$.

(4) Let $\mathcal{T}_h$ be the mesh composed of the cells $[ih, (i+1)h]$, $i \in \{0{:}I\}$, with uniform mesh size $h = \frac{1}{I+1}$. Let $P(\mathcal{T}_h)$ be the finite element

space composed of continuous piecewise linear functions that are zero at 0 and 1. Write the discrete problem in $P(\mathcal{T}_h)$.

(5) Derive an error estimate in $H^1(D)$. (Note that we only have $u \in H^{r_{\max}}(D)$ for some $r_{\max} \in (1,2)$ since f is not in $L^2(D)$.)

(6) Derive an improved error estimate in $L^2(D)$. (*Hint:* Use a duality argument. Consider the problem $-\partial_x((1+\sin(x)^2)\partial_x s) = v$ in D with $s(0) = s(1) = 0$ and $v \in L^2(D)$. Accept as a fact that s is in $H^2(D)$ if $v \in L^2(D)$, and there is $c > 0$ such that $|s|_{H^2} \leq c\|v\|_{L^2}$ for all $v \in L^2(D)$.)

(7) Bonus question if you have time. Prove the elliptic regularity statement in the above hint.