Applied/Numerical Analysis Qualifying Exam

January 8, 2012

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
January 8, 2012

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.

(b) Consider a function $f \in C^{(1)}(\mathbb{R})$ such that both f and f' are in $L^1(\mathbb{R})$, and $\int_{\mathbb{R}} f(x)dx = 1$. Define the sequence of functions $\{T_n(x) := n^2 f'(nx) \colon n = 1, 2, \ldots\}$. Show that, in the sense of distributions — i.e., in $\mathcal{D}'$ —, T_n converges to δ'.

Problem 2. Let $M : C[0,1] \to C[0,1]$ be defined by $M(u) := \int_0^1 (2 + st + u(s)^2)^{-1} ds$. Let $\|\cdot\| := \|\cdot\|_{C[0,1]}$. Let $B_r := \{u \in C[0,1] \mid \|u\| \le r\}$.

(a) Show that $M : B_1 \to B_{1/2} \subset B_1$.

(b) Show that M is Lipschitz continuous on B_1, with Lipschitz constant $0 < \alpha < 1$ – i.e., $\|M[u] - M[v]\| \le \alpha \|u - v\|$.

(c) Show that M has a fixed point in B_1. State the theorem you are using to show that the fixed point exists.

Problem 3. Let $Lu = -\frac{d^2u}{dx^2}$, $-\pi \le x \le \pi$, with the domain of L given by

$$D_L := \{u \in L^2[-\pi,\pi] : u'' \in L^2[\pi,\pi],\ u(-\pi) = -u(\pi),\ u'(-\pi) = -u'(\pi)\}.$$

(a) Show that L is self adjoint on $D(L)$.

(b) Find the Green's function $G(x,y)$ for the problem $Lu = f$, $u \in D_L$.

(c) Show that $Ku := \int_{-\pi}^{\pi} G(\cdot,y)u(y)dy$ is a compact self-adjoint operator.

(d) Without actually finding them, show that the eigenfunctions of L form a complete, orthogonal set for $L^2[-\pi,\pi]$. (Hint: Relate the eigenfunctions of L to those of K. Use compactness.)

Problem 4. Let T be a (possibly unbounded) linear operator on a Hilbert space $\mathcal{H}$, defined on the domain D_T.

(a) Define these: the resolvent set of T, $\rho(T)$; the discrete spectrum, $\sigma_d(T)$; the continuous spectrum, $\sigma_c(T)$; and the residual spectrum, $\sigma_r(T)$.

(b) Assume T is bounded. Show that the set $\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\} \subseteq \rho(T)$. (Hint: Use a Neumann series expansion.)

(c) Let $\mathcal{H} = \ell^2$, with the usual inner product. Define T to be the shift operator

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Show that every $|\lambda| > 1$ is in $\rho(T)$, that $\lambda = 1$ is in $\sigma_c(T)$, and that $\lambda = 0$ is in $\sigma_r(T)$.

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Numerical analysis part
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Problem 1:

Let $\Omega = (0,1) \times (0,1)$, $f \in C^0(\overline{\Omega})$ and $q \in \mathbb{R}$ with $q \geq 0$. Consider the boundary value problem

$$-\Delta u + qu = f \qquad \text{in } \Omega; \qquad u = 0 \qquad \text{on } \partial\Omega.$$

We are interested in approximating the quantity $\alpha := \int_{\partial\Omega} \mathbf{n} \cdot \nabla u$ where $\mathbf{n}$ is the outward unit normal of Ω.

1. The boundary problem has a weak formulation: Find $u \in \mathbb{V}$ such that

$$\forall v \in \mathbb{V} : \qquad a(u,v) = L(v).$$

Identify $\mathbb{V}$, $a(u,v)$ and $L(v)$. Show that there exists a unique solution $u \in \mathbb{V}$ satisfying the above weak formulation.

2. Let $\{\mathcal{T}_h\}_{0<h<1}$ be a sequence of conforming shape-regular subdivisions of Ω such that $\mathrm{diam}(T) \leq h$, for all $T \in \mathcal{T}_h$ and define

$$\mathbb{V}_h := \left\{ v \in C^0(\overline{\Omega}) \cap \mathbb{V} \quad | \quad \forall T \in \mathcal{T}_h, \quad v|_T \quad \text{is linear} \right\}.$$

Write the weak formulation satisfied by the finite element approximation $u_h \in \mathbb{V}_h$ of u. Prove that the function u_h exists and is unique.

3. Assume from now that $u \in H^2(\Omega)$. Derive the error estimate

$$\|u - u_h\|_{H^1(\Omega)} \leq c_1 h \|u\|_{H^2(\Omega)},$$

where c_1 is a constant independent of h and u.
Hint: you can use without proof the fact that there exists a constant C independent of h such that for any $v \in H^2(\Omega)$

$$\inf_{v_h \in \mathbb{V}_h} \|v - v_h\|_{\mathbb{V}} \leq Ch\|v\|_{H^2(\Omega)}.$$

4. Show that that for the constant function $w(\mathbf{x}) = 1$ we have

$$\alpha = a(u,w) - L(w).$$

Now let $\alpha_h := a(u_h, w) - L(w)$. Using the previous parts, show that when $q > 0$ there holds

$$|\alpha - \alpha_h| \leq c_2 h^2 \|u\|_{H^2(\Omega)},$$

where c_2 is a constant independent of h and u. What can you say about $|\alpha - \alpha_h|$ when $q = 0$?

Problem 2:

Let K be a polyhedron in $\mathbb{R}^d$, $d \geq 1$. Let $h = \mathrm{diam}(K)$ and define

$$\hat{K} = \{\hat{\mathbf{x}} = \mathbf{x}/\mathrm{diam}(K), \quad \mathbf{x} \in K\}.$$

Show that there exists a constant c solely depending on $\hat{K}$ such that for any $v \in H^1(K)$,

$$\|v\|_{L^2(\partial K)} \leq c \left(h^{-1/2} \|v\|_{L^2(K)} + h^{1/2} \|\nabla v\|_{L^2(K)} \right).$$

Problem 3:

Let $u_0 : (0,1) \to \mathbb{R}$ be a given smooth initial condition and $T > 0$ be a given final time. Let $u : [0,T] \times \Omega \to \mathbb{R}$ be a smooth function satisfying $u(t,0) = u(t,1) = 0$ for any $t \in [0,T]$ and

$$\forall v \in C_c^\infty([0,T) \times (0,1)) :$$

$$-\int_0^T \int_0^1 u(t,x)\ v_t(t,x)\ dx\ dt - \int_0^1 u_0(x) v(0,x)\ dx + \int_0^T \int_0^1 u_x(t,x)\ v_x(t,x)\ dx\ dt + \int_0^T \int_0^1 u(t,x)\ v(t,x)\ dx\ dt = 0. \tag{4.1}$$

Here $C_c^\infty([0,T) \times (0,1))$ is the space of functions belonging to $C^\infty([0,T] \times [0,1])$ and compactly supported in $[0,T) \times (0,1)$.

1. Derive the corresponding strong formulation.
2. Let $N > 0$ be an integer, $h = 1/N$ and $x_n = n\ h$, $n = 0, ..., N$. Derive the semi-discrete approximation of (4.1) using continuous piecewise linear finite elements.
3. In addition, let $M > 0$ be an integer, $\tau = T/M$ and $t_m = m\tau$ for $m = 0, .., M$. Write the fully discrete schemes corresponding to backward Euler and forward Euler methods, respectively.
4. Prove that the backward (implicit) Euler scheme is unconditionally stable while the forward (explicit) Euler method is stable provided $\tau \leq ch^2$, where c is a constant independent of h and τ.