Applied/Numerical Analysis Qualifying Exam

January 8, 2013

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name ____

Combined Applied Analysis/Numerical Analysis Qualifier

Applied Analysis Part

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Instructions: *Do all problems in this part of the exam. Show all of your work clearly.*

1. The eigenvalues of the given symmetric matrix A can be ordered

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5.$$

Use the Courant Minimax Principle to find the value for λ_3.

$$A = \begin{pmatrix} 5 & 12 & -3 & 6 & 2 \\ 12 & 2 & 0 & -1 & 0 \\ -3 & 0 & 2 & 1 & 0 \\ 6 & -1 & 1 & 13 & 7 \\ 2 & 0 & 0 & 7 & 2 \end{pmatrix}.$$

2. Answer the following:

a. State the Weierstrass Approximation Theorem for functions defined on the interval $[0, 1]$.

b. Given that $C([0, 1])$ is dense in $L^2([0, 1])$, prove that the set of functions $\{x^{3n}\}_{n=0}^{\infty}$ is dense in $L^2([0, 1])$.

c. Explain how you would produce a complete orthonormal set from the functions $\{x^{3n}\}_{n=0}^{\infty}$, and prove that your orthonormal set is complete in $L^2([0, 1])$.

3. Let $H = \ell^2$ and suppose $L : H \to H$ is the right-shift operator so that for $u \in H$

$$\begin{aligned} (Lu)_1 &= 0 \\ (Lu)_n &= u_{n-1}, \quad n = 2, 3, \ldots \end{aligned}$$

a. Show that L is a bounded, linear operator and compute $\|L\|$ (not just an upper bound).

b. Find the adjoint L^* for this operator.

c. Show that if $|\lambda| \geq 1$ the closure of the range of $L - \lambda I$ is H.

4. Suppose H is a Hilbert space and $K : H \to H$ is a compact linear operator.

a. Prove that K^*K is a self-adjoint, compact operator, and that the eigenvalues of K^*K are all non-negative.

b. Prove that there exist positive numbers $\{\alpha_i\}_{i=1}^{N}$ and orthonormal sets $\{\phi_i\}_{i=1}^{N}$ and $\{\psi_i\}_{i=1}^{N}$ (where N may be either a positive integer or ∞) so that

$$Ku = \sum_{i=1}^{N} \alpha_i \langle u, \phi_i \rangle \psi_i$$

for all $u \in H$.

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Cover Sheet – Numerical Analysis Part

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In all questions below, you may use standard estimates for finite element interpolation operators without proving them.

Problem 1. (a) You may assume the inequality

$$\|u\|^2_{H^1(\hat\tau)} \le C\bigg(\int_{\hat\tau} |\nabla u|^2\, d\hat x + \bar u^2\bigg), \qquad \text{for all } u \in H^1(\hat\tau).$$

Here $\hat\tau$ is the reference triangle in $\mathbb{R}^2$, $\bar u$ denotes the mean value of u on $\hat\tau$ and $\mathbb{P}^k$ denotes the polynomials of (x, y) of degree at most k. Let τ denote a general triangle in $\mathbb{R}^2$. Show that

$$\|u\|^2_{H^1(\tau)} \le C_\theta\bigg\{\int_\tau |\nabla u|^2\, dx + h^2\bar u^2\bigg\}, \qquad \text{for all } u \in \mathbb{P}^1.$$

Here θ denotes the minimum angle of τ and h its diameter. Now $\bar u$ denotes the mean value of u on τ. (You may assume, without proof, standard properties involving the dependence on θ of the affine map of $\hat\tau$ onto τ.)

(b) Let V_h be the space of continuous piecewise linear functions with respect to a quasi-uniform mesh $\Omega = \cup_{i=1}^N \tau_i$. Consider the one point quadrature approximation

$$Q_{\tau_i}(g) := |\tau_i|\, g(b_i) \approx \int_{\tau_i} g,$$

where $|\tau_i|$ is the area of τ_i and b_i is its barycenter.

Consider the finite element problem: Find $u_h \in V_h$ satisfying

$$A_h(u_h, \phi) = F_h(\phi), \qquad \text{for all } \phi \in V_h.$$

Here for $u_h, v_h \in V_h,$, A_h and F_h are given by

$$A(u_h, v_h) := \sum_{i=1}^N \big(Q_{\tau_i}(\nabla u_h \cdot \nabla v_h) + Q_{\tau_i}(u_h v_h)\big) \quad \text{and} \quad F_h(v_h) := \sum_{i=1}^N Q_{\tau_i}(f v_h).$$

respectively. Show that

$$Q_{\tau_i}(|\nabla u|^2) = \int_{\tau_i} |\nabla u|^2 \qquad \text{and} \qquad Q_{\tau_i}(|u|^2) = |\tau_i|\,\bar u^2, \qquad \text{for all } u \in \mathbb{P}^1.$$

(c) Use Parts (b) and (c) above to show that the form $A_h(\cdot,\cdot)$ is V_h-elliptic, i.e.,

$$A_h(v_h, v_h) \ge c\|v_h\|^2_{H^1(\Omega)}, \qquad \text{for all } v_h \in V_h,$$

holds with c independent of h.

Problem 2. Let Ω be a convex polygonal domain of $\mathbb{R}^2$. Given $f \in L^2(\Omega)$, we denote by $u \in H^1_0(\Omega)$ the solution of the Poisson problem:

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

We note that u satisfies full elliptic regularity, i.e., $u \in H^2(\Omega)$.

We consider a *non conforming* finite element method to approximate u. Let $\{\mathcal{T}_h\}_{0<h<1}$ be a sequence of conforming shape regular subdivisions of Ω such that $\operatorname{diam}(T) \le h$. Denote by X_h the spaces of continuous, piecewise linear polynomials subordinate to the subdivisions $\mathcal{T}_h$, $0 < h < 1$.

The numerical method consists of finding $u_h \in X_h$ such that for all $v_h \in X_h$:

$$a_h(u_h, v_h) := \int_\Omega \nabla u_h \cdot \nabla v_h - \int_{\partial\Omega} \partial_\nu u_h \, v_h + \frac{\alpha}{h} \int_{\partial\Omega} u_h \, v_h = \int_\Omega f \, v_h.$$

Here ν denotes the outward pointing unit normal (defined almost everywhere), $\partial_\nu u := \nabla u \cdot \nu$ and $\alpha > 0$ is a constant yet to be determined. Note that $X_h \not\subset H_0^1(\Omega)$ but $X_h \subset H^1(\Omega)$.

(a) Explain why $a_h(u, v_h)$ makes sense for any $v_h \in X_h$ and show Galerkin orthogonality, i.e.,

$$a_h(u - u_h, v_h) = 0, \quad \text{for all } v_h \in X_h.$$

(b) For any $v_h \in X_h$, defined the mesh dependent norm

$$\|v_h\|_h := \left(\|\nabla v_h\|^2_{L_2(\Omega)} + \frac{\alpha}{h} \|v_h\|^2_{L_2(\partial\Omega)} \right)^{1/2}.$$

Show that there exists a constant c_0 independent of h such that for all $v_h \in X_h$

$$\int_{\partial\Omega} |\nabla v_h|^2 \le \frac{c_0}{h} \int_\Omega |\nabla v_h|^2.$$

Using this fact, deduce that for all $v_h \in \mathbb{X}_h$,

$$a_h(v_h, v_h) \ge \frac{1}{2} \|v_h\|_h^2,$$

provided $\alpha \ge c_0$.

(c) Let I_h denote the Lagrange finite element interpolation operator associated with X_h. You may use the following estimate without proof: For $i = 1, 2$,

$$\left\| \frac{\partial(u - I_h u)}{\partial x_i} \right\|_{L^2(e)} \le C h^{1/2} \|u\|_{H^2(\tau)}.$$

Take $\alpha = c_0$ and derive an optimal error estimate for $\|u - u_h\|_h$.

Problem 3. Given the boundary value problem: find $u(x, t)$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2} - b(x) \frac{\partial u}{\partial x} + f(x), \ 0 < x < 1, \ 0 < t \le T, \\ & u(0, t) = 0, \ u(1, t) = 0, \ 0 < t \le T \\ & u(x, 0) = v(x), \ 0 \le x \le 1, \end{aligned}$$

where $\kappa = \mathit{const} > 0$, $b(x) \in C^0[0, 1]$, $v(x)$, and $f(x)$ are given smooth functions. Let $x_i = ih$ with $h = 1/N$ and $t_n = n\tau$, with $n = 0, 1, \ldots, J$ and (time step size) $\tau = T/J$.

(1) Write down a forward (explicit) Euler fully discrete scheme for the above problem based on a finite difference discretization in space which upwinds the $b(x)$ term.

(2) Find a Courant (CFL) condition and show that if this condition is satisfied,

$$\|U^{n+1}\|_\infty \le \|U^n\|_\infty + \tau \|f(t_n)\|_\infty.$$

Here U^n is the approximation at t_n of part (a).

(3) Define the fully discrete method but with backward (implicit) Euler time stepping and show that this scheme is unconditionally stable, i.e., prove that for any positive τ,

$$\|U^{n+1}\|_\infty \le \|U^n\|_\infty + \tau \|f(t_{n+1})\|_\infty$$

holds.