Complex Analysis
Qualifying Examination

August 2016

1. State the following three theorems, with precise hypotheses and conclusions: Morera's theorem, the Weierstrass factorization theorem for entire functions, and Hurwitz's theorem about limits of sequences of holomorphic functions.

2. Suppose that g is a nonconstant entire function, and $u \colon \mathbb{C} \to \mathbb{R}$ is a twice continuously differentiable function. Prove that if the composite function $u \circ g$ is harmonic, then u is harmonic.

3. Give an example of a holomorphic function f on $\mathbb{C} \setminus \{ x + iy : x \leq 0 \text{ and } y = 0 \}$, the plane with a slit along the negative part of the real axis, such that the limit from above $\lim_{y \to 0^+} f(-1 + iy)$ exists but the limit from below $\lim_{y \to 0^-} f(-1 + iy)$ does not exist.

4. Prove that if $\operatorname{Re}(z) \leq 0$ and $\operatorname{Re}(w) \leq 0$, then $|e^z - e^w| \leq |z - w|$.

5. Prove normality of the family of holomorphic functions f in the unit disk such that
$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta < 2016.$$

6. Apply the residue theorem to show that $\displaystyle\int_0^\infty \frac{(\log x)^2}{1 + x^2} \, dx = \frac{\pi^3}{8}$.

7. Prove that if $f : \mathbb{C} \setminus \{0\} \to G$ is a biholomorphic mapping from the punctured plane onto an open subset G of $\mathbb{C}$, then G must be $\mathbb{C} \setminus \{b\}$ for some complex number b.

8. Suppose f is a holomorphic function on the upper half-plane such that $f(z)$ is never equal to $\pm i$. Prove the existence of a holomorphic function g on the upper half-plane such that $f(z) = \tan(g(z))$ for every z.

9. How many zeroes does the function $z^8 + \exp(2016\pi z)$ have in the region where $\operatorname{Re}(z) < 0$ (the left-hand half-plane)? Explain.

10. State and prove one of the following inequalities concerning functions on disks: the Schwarz lemma for holomorphic functions, Harnack's inequality about positive harmonic functions, Cauchy's estimate for derivatives of holomorphic functions.