Complex analysis qualifying exam, August 2018.

1. Give the statements of the following results:

   (a) Morera's theorem;
   (b) Schwarz's lemma;
   (c) Runge's theorem.

2. How many zeros of the polynomial

$$z^4 + 3z^2 + z + 1$$

lie in the right half-plane?

3. Let f be holomorphic in a complex domain containing the unit disk. Suppose that f has a pole at $z = 1$. Prove that then the Taylor series of f at $a = 0$ diverges at every point z on the unit circle.

4. Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

be a complex polynomial whose roots lie in the lower half-plane. Consider the polynomials

$$\alpha(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_1 z + \alpha_0, \quad \beta(z) = \beta_n z^n + \beta_{n-1} z^{n-1} + ... + \beta_1 z + \beta_0,$$

where $\alpha_k = \text{Re } a_k$ and $\beta_k = \text{Im } a_k$ for $k = 0, 1, ..., n$. Show that the polynomials $\alpha(z)$ and $\beta(z)$ have only real roots.

5. Find a conformal mapping which maps the strip $\{0 < \text{Re } z < 1\}$ onto the disk with a slit $\{|z| < 1, \ z \notin [0, 1]\}$.

6. Calculate the integral

$$\int_0^{2\pi} \frac{d\theta}{(5 + 4\cos\theta)^2}.$$

7. Let f be a non-constant holomorphic function in a neighborhood of the closed unit disk such that $|f| = 1$ on the unit circle. Show that f is a finite Blaschke product (a product of finitely many Möbius transforms of the unit disk).

8. Show that there exists a holomorphic function in $\{|z| > 4\}$ whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)}$$

and prove that there is no holomorphic function in the same domain with the derivative

$$\frac{z^2}{(z-1)(z-2)(z-3)}.$$

9. Let u and v be non-constant harmonic functions in a complex domain Ω. Suppose that uv is also harmonic in Ω. Prove that there exists a real constant c such that $u + icv$ is holomorphic in Ω.

10. Let f be an entire function of exponential type such that $f(\sqrt{n}) = n$ for any positive integer n. Find $f(i)$. Explain why the value you found is the only possible.