Complex analysis qualifying exam, January 2010.

1. Give the statements of the following theorems:

(a) Montel's theorem;
(b) The Weierstrass factorization theorem.

2. Suppose that a function f is holomorphic in $\{0 < |z - a| < r\}$ for some $r > 0, a \in \mathbb{C}$ and that f'/f has a pole of order one at a. Prove that then f has a pole or a zero at a.

3. Prove that all zeros of the function $\tan z - z$ are real.

4. Let $\{f_n\}$ be a sequence of holomorphic functions in a complex domain Ω. Suppose that $f_n(a)$ converges for some $a \in \Omega$ and that the functions $\Re f_n$ converge normally in Ω. Prove that then f_n converge normally in Ω.

5. Let $f_1, f_2, ..., f_n$ be holomorphic in a bounded complex domain Ω and continuous in the closure of Ω. Let $g = |f_1| + |f_2| + ... + |f_n|$.

a) Prove that the maximum of g is attained on the boundary of Ω.

b) Prove that if $g \equiv$ const in Ω then all f_k are constants.

6. Let f be a function holomorphic in the unit disk $\mathbb{D}$ and continuous in the closure $\bar{\mathbb{D}}$.

a) Show that if $\Re f = 0$ on ∂D then f is a constant.

b) Show that the previous statement becomes false if $\partial\mathbb{D}$ is replaced with a proper subarc of $\partial\mathbb{D}$.

7. Let entire functions f and g satisfy $e^f + e^g \equiv 1$. Prove that then both are constants.

8. Find a general formula for all functions $w(z)$ that map the domain $\Omega = \{|z| < 1\} \setminus [1/2, 1]$ conformally onto the domain $\{|\Im z| < 1\}$.

9. Let u be a real-valued harmonic function in $\mathbb{C} \setminus \{0\}$. Show that then

$$u(z) = c \log |z| + \Re f(z)$$

for some real constant c and a function f holomorphic in $\mathbb{C} \setminus \{0\}$.

10. Prove that for a function $f : \mathbb{C} \to \hat{\mathbb{C}}$ it holds that

$$res_{z=a} f = -res_{z=-a} f$$

if f is even and that

$$res_{z=a} f = res_{z=-a} f$$

if f is odd. We assume that all the residues are correctly defined, i.e. f is holomorphic in a punctured neighborhood of a.