Complex analysis qualifying exam, January 2012.

1. Give the statements of the following:
   (a) The Mittag-Leffler theorem;
   (b) Harnack's lemma.

2. Consider an infinite product
$$\pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$
   (a) Prove that the product converges normally in $\mathbb{C}$.
   (b) Find the elementary entire function that the product converges to (prove your answer).
   (c) Use the previous parts to prove Wallis' formula:
$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot ...$$

3. Let $F = \{f_a\}_{a \in A}$ be a family of functions holomorphic in a neighborhood of the closed unit disk $\bar{\mathbb{D}} = \{|z| \le 1\}$. Suppose that
$$\int_0^{2\pi} |f_a(e^{i\phi})|^{1/2} d\phi \le 1$$
for any $a \in A$. Prove that F is a normal family in the unit disk $\mathbb{D} = \{|z| < 1\}$.

4. Let γ be a closed curve in the right half-plane that has index N with respect to the point 1. Find
$$\int_\gamma e^{\frac{1}{z^2-1}} \sin(\pi z) dz.$$

5. Let $\Omega \neq \mathbb{C}$ be a simply-connected complex domain containing a point c. Let $\phi : \Omega \to \mathbb{D}$ be a conformal mapping such that $\phi(c) = 0$. The function $g_c(z) = \log|\phi(z)|$ is called the Green function of Ω corresponding to c. Prove that $g_a(b) = g_b(a)$ for any $a, b \in \Omega$.

6. Write a formula for a conformal map from the upper half-plane to $\{z |\ \Re z > 0,\ |\Im z| < 1\}$.

7. Let F be an entire function such that
$$|F(z)| \le e^{|z|^\lambda}$$
for some $\lambda > 0$ and large enough $|z|$. Let $F(z) = \sum_0^\infty a_n z^n$ for all $z \in \mathbb{C}$. Prove that then
$$|a_n| \le \left(\frac{e\lambda}{n}\right)^{n/\lambda}$$
for large enough n.

8. Let F be a function holomorphic and bounded in the upper half-plane $\mathbb{C}_+$. Suppose that F has period 1 ($F(z+1) = F(z)$ for all $z \in \mathbb{C}_+$). Prove that $F(z)$ has a finite limit as $\Im z \to +\infty$.

9. Let $u(z)$ be a bounded harmonic function in $\mathbb{D}$ such that the limit
$$\lim_{r \to 1-} u(re^{i\phi})$$
is equal to 1 for $0 < \phi < \pi$ and to 0 for $\pi < \phi < 2\pi$. Find $u(1/2)$.

10. Let F be an entire function. We say that $a \in \mathbb{C} \cup \{\infty\}$ is an asymptotic value for F if there exists a continuous curve going from a finite point to infinity such that F tends to a along that curve. Prove that for any non-constant entire function ∞ is an asymptotic value.