Complex Analysis Qualifying Examination

January 2013

1. Suppose $\sum_{n=0}^{\infty} a_n z^n$ is the Maclaurin series of the rational function $\dfrac{z}{1-z-z^2}$. Prove that the coefficient sequence $\{a_n\}_{n=0}^{\infty}$ is the sequence of Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ... (defined by the property that each number is the sum of the preceding two).

2. When f and g are functions on the real line, the convolution $f * g$ is defined as follows:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,\mathrm{d}t.$$

   Prove that if $f(x) = \dfrac{1}{x^2+1}$ and $g(x) = \dfrac{1}{x^2+4}$, then $(f * g)(x) = \dfrac{3\pi/2}{x^2+9}$.

3. Suppose that $a_0 > a_1 > \cdots > a_{2013} > 0$. Prove that $\displaystyle\sum_{n=0}^{2013} a_n z^n \neq 0$ when $|z| \leq 1$.

4. Suppose f is a biholomorphic map from the unit disk onto the horizontal strip where $-1 < \operatorname{Im}(z) < 1$, normalized such that $f(0) = 0$. Determine the value of $|f'(0)|$.

5. Let $\mathcal{F}$ denote the family of power series $\sum_{n=1}^{\infty} a_n z^n$ for which $|a_n| \leq n$ for every positive integer n. Is $\mathcal{F}$ a normal family in the open unit disk? Explain why or why not.

6. Suppose f is a holomorphic function on $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$, the punctured unit disk. Prove that if $\operatorname{Re}(f)$ is a bounded function, then f has a removable singularity at 0.

7. Prove that every complex number is in the range of the entire function $e^{3z} + e^{2z}$.

8. Suppose f is holomorphic in the unit disk, and

$$f(2z) = 2f(z)f'(z) \qquad \text{whenever } |z| < 1/2.$$

   Prove that f is the restriction to the unit disk of some entire function.

9. Prove that if f is a holomorphic function in the unit disk such that $|f|$ is harmonic, then f must be constant.

10. State and prove *one* of the following theorems: Hadamard's three-circles theorem, the monodromy theorem, Hurwitz's theorem.