Complex analysis qualifying exam, January 2014.

1. Give the statements of the following results:

(a) Montel's theorem;
(b) Harnack's lemma;
(c) Mittag-Leffler's theorem.

2. Let $f(z)$ be analytic in $\Omega = \{|z| > 1\}$. Suppose that f satisfies $|f(z)| < |z|^n$ for all $z \in \Omega$ and for some $n > 0$. Prove that either f has finitely many zeros in $\{|z| > 2\}$ or f is identically zero.

3. Let f be an entire function that is not a polynomial. Denote

$$M(r) = \max_{|z|=r} |f(z)|.$$

Show that

$$\lim_{r \to \infty} \frac{M(r/2)}{M(r)} = 0.$$

4. Let f and g be analytic functions in the same connected complex domain Ω. Suppose that $|f| = \Re g$ in Ω. Show that f and g are constants.

5. Consider the line in the z plane defined by the following equation:

$$3\Re(z) + 4\Im(z) = 5.$$

Under the inversion that sends z to $1/z$, this line transforms into a circle. Find the center and the radius of that circle.

6. Consider a rational function $f(z) = q(z)/p(z)$, where p is a polynomial of degree n and q is a polynomial of degree $n-2$ or less. If $z_1, z_2, ..., z_n$ are distinct roots of p, prove that the residues of f satisfy

$$\sum_{k=1}^{n} Res(f, z_k) = 0.$$

7. Let f be an entire function. Prove that all the coefficients in the power series expansion of f at the origin are real if and only if f is real on the real line.

8. Find a biholomorphic map between the unit disk and the parabolic region in the z plane defined by the property that $\Im(z) > (\Re(z))^2$.

9. Use harmonic functions to prove the following statement: For any continuous function f on the unit circle $\mathbb{T} = \{|z| = 1\}$ there exists a sequence of polynomials $p_n(z, \bar{z})$ of z and $\bar{z}$ that converges to f uniformly on $\mathbb{T}$ (The Weierstrass Approximation Theorem for the unit circle).

10. Show that there is no entire function of finite order, except the zero function, that has roots at all points z such that $\exp(\exp z) = 1$.