Complex Analysis
Qualifying Examination

January 2015

1. Prove that if z is a complex number and k is a positive integer, then
$$\left|\operatorname{Im}(z^k)\right| \leq k\,|\operatorname{Im}(z)|\;|z|^{k-1}.$$

2. There is a holomorphic function f such that $f(z)e^{f(z)} = z$ for every z in a neighborhood of the origin. Find the first three nonzero terms in the Maclaurin series of f.

3. Prove that $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{\cos\theta}\,d\theta = \sum_{n=0}^{\infty} \frac{1}{(n!\,2^n)^2}$.

4. The following proposition is a special case of Jensen's generalization of the Schwarz lemma: *If f is a holomorphic function that maps the open unit disk into itself, and if z_1 and z_2 are two zeroes of f in the unit disk, then*
$$|f(z)| \leq \left|\frac{(z-z_1)(z-z_2)}{(1-\overline{z}_1 z)(1-\overline{z}_2 z)}\right| \qquad \text{when } |z| < 1.$$
Prove this proposition.

5. Suppose f is an entire function such that the product $|\operatorname{Re} f|\,|\operatorname{Im} f|$ is bounded. Prove that f must be a constant function.

6. Find a surjective holomorphic mapping from $\{\, z \in \mathbb{C} : |z| < 1 \,\}$ (the open unit disk) onto $\{\, z \in \mathbb{C} : 1 < |z| \,\}$ (the complement of the closed unit disk). [Notice that the inversion $1/z$ is not holomorphic at the origin. Moreover, the solution to this problem cannot possibly be an injective function.]

7. Suppose $u\colon \mathbb{C}\setminus\{0\} \to \mathbb{R}$ is a nonconstant, real-valued harmonic function on the punctured plane. Prove that the image is all of $\mathbb{R}$.

8. Suppose f has a simple pole at the origin, and g denotes $1/f$ (the reciprocal function). How is the residue at the origin of the composite function $f \circ g$ related to the residue at the origin of f?

9. Let U denote $\{\, z \in \mathbb{C} : \operatorname{Im} z > 0 \,\}$, the open upper half-plane, and let f denote the restriction to U of the principal branch of the logarithm. Consider the sequence of iterates f, $f \circ f$, $f \circ f \circ f$, ... Is this sequence of functions locally bounded in U? Explain.

10. State two of the following three theorems: the monodromy theorem, Runge's approximation theorem, Hadamard's three-circles theorem.