Complex Analysis
Qualifying Examination

January 2016

1. State the following three theorems, with precise hypotheses and conclusions: the Schwarz reflection principle, Runge's theorem about polynomial approximation of holomorphic functions, and Mittag-Leffler's theorem about meromorphic functions with prescribed poles.

2. Suppose f is a holomorphic function on a connected open set, and $u = \operatorname{Re}(f)$. Prove that if the product $u\overline{f}$ is holomorphic, then f must be a constant function.

3. Suppose γ is a simple, closed, continuously differentiable curve. What are all the possible values of $\frac{1}{2\pi i}\int_\gamma \frac{z}{z^2+1}\,dz$ for different choices of γ? Explain.

4. An inequality from real calculus says that if $x \in \mathbb{R}$ and $|x| \le \frac{\pi}{2}$, then $\frac{2}{\pi}|x| \le |\sin(x)|$. Prove that this inequality extends to complex numbers: namely, if $z \in \mathbb{C}$ and $|z| \le \frac{\pi}{2}$, then $\frac{2}{\pi}|z| \le |\sin(z)|$.

5. A map is called *proper* when the inverse image of every compact set is compact. Prove that there does *not* exist a surjective proper holomorphic map $f : \mathbb{D} \to \mathbb{C}$, where $\mathbb{D}$ denotes $\{\, z \in \mathbb{C} : |z| < 1 \,\}$, the open unit disk.

6. Give an example of a nonpolynomial entire function f such that the range of f is all of $\mathbb{C}$, but the range of f', the derivative, is not all of $\mathbb{C}$.

7. Does there exist a holomorphic function f on the region $\{\, z \in \mathbb{C} : |z| > 1 \,\}$ (the exterior of the unit disk) such that $(f(z))^{2016} = z + 1$ for every point z in the region? Explain.

8. Suppose f is a holomorphic function on $\{\, z \in \mathbb{C} : |z| < 1 \,\}$, the open unit disk, with the property that $\operatorname{Re} f(z) > 0$ for every point z in the disk. Prove that $|f'(0)| \le 2 \operatorname{Re} f(0)$.

9. Prove that $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{1+\sin^2\theta}\,d\theta = \frac{\pi}{\sqrt{2}}$.

10. Suppose circles of radius r and radius s are externally tangent at the point $1/2$ and internally tangent to the unit circle. There are infinitely many such configurations, two of which are illustrated in the diagram below. Prove that $\frac{1}{r} + \frac{1}{s} = \frac{16}{3}$ for every such configuration.