Complex Analysis
Qualifying Examination

7 January 2019

1. State (a) Runge's theorem about polynomial approximation, (b) Mittag-Leffler's theorem about prescribed singularities, and (c) Picard's great theorem.

2. Suppose $f(z)$ has an essential singularity when $z = 0$, and $g(z)$ has an essential singularity when $z = 0$. Prove that at least one of the functions $f(z) + g(z)$ and $f(z)g(z)$ has an essential singularity when $z = 0$.

3. Suppose f is holomorphic on $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk), and $|f(z)| < 1$ when $|z| < 1$. How large can $|f'(1/7)|$ be?

4. Prove that on the region $\mathbb{C} \setminus \{x + 0i : x \in \mathbb{R} \text{ and } |x| \leq 1 \}$ (the plane with a slit along the real axis from -1 to 1), there exists a holomorphic function $f(z)$ such that $f'(z) = \dfrac{1}{1 - z^2}$, but there does not exist a holomorphic function $g(z)$ such that $g'(z) = \dfrac{z}{1 - z^2}$.

5. Prove there are infinitely many values of the complex variable z for which $\sin(z) = \sin(iz)$.

6. Prove that if $t \in \mathbb{R}$, then $\displaystyle\lim_{t \to 0} \int_{-\infty}^{\infty} \frac{x \sin(tx)}{1 + x^2}\, dx = \pi$.

7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a nonconstant entire function. Which of the following sets must be countably infinite? (a) $f(\mathbb{Z})$ (b) $f(\mathbb{Q})$ (c) $f^{-1}(\mathbb{Z})$ (d) $f^{-1}(\mathbb{Q})$
   Explain why.

8. Suppose f is an entire function, and suppose the sequence of derivatives f', f'', $f^{(3)}$, … converges uniformly on compact sets to a limit function that is not identically zero. Prove the existence of a natural number N such that $f^{(n)}(z) \neq 0$ when $|z| < 1$ and $n > N$.

9. Either construct or prove the existence of a biholomorphic mapping (an analytic bijection) from $\mathbb{C} \setminus \{ x + 0i : x \in \mathbb{R} \text{ and } |x| \leq 1 \}$ (the plane with a slit along the real axis from -1 to 1) onto $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$ (the punctured unit disk).

10. Suppose f is an entire function with the property that $f(2z) = \dfrac{f(z) + f(z+1)}{2}$ for all z. Prove that f must be a constant function.