Complex Analysis
Qualifying Examination

January 2022

1. Prove the following formula obtained in 1776 by Charles Hutton:

$$\frac{\pi}{4} = 2\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right).$$

2. Show that there are exactly two values of the complex variable z for which $\cos(z) = 2$ and $|z| < 2$.

3. Prove that if f is analytic in the punctured disk $\{\, z \in \mathbb{C} : 0 < |z| < 1 \,\}$, and f has a pole at 0, then the function e^f has an essential singularity at 0.

4. Suppose that $u\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic, and $u(0,0) = 0$. Show that for every positive ε, there exists a point (x, y) such that $0 < x^2 + y^2 < \varepsilon$ and $u(x, y) = 0$. In other words, a harmonic function cannot have an isolated zero.

5. Prove that if f is a nonconstant entire function, then

$$\max_{|z|=1} \left| f(z) - \frac{1}{z} \right| > 1.$$

6. Determine every entire function f satisfying the property that $|f(z)| = f(|z|)$ for all values of the complex variable z.

7. When n is a positive integer and $z \in \mathbb{C}$, let $f_n(z)$ denote $(\cos(z^n))^n$. Is the sequence $\{f_n\}_{n=1}^{\infty}$ a normal family in the open unit disk? Explain.

8. Construct a biholomorphic mapping from the half-infinite strip

$$\{\, z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1 \text{ and } 0 < \mathrm{Im}(z) \,\}$$

onto the unit disk.

9. Suppose f is an analytic function in the right-hand half-plane, the value $f(2)$ is positive, and $(f(z))^2 + 2f(z) = z^2$ when $\mathrm{Re}(z) > 0$. If f is continued analytically along the semicircle $2e^{it}$, $0 \le t \le \pi$, then what is the value $f(-2)$? Explain.

10. State Runge's theorem on polynomial approximation, the Weierstrass factorization theorem for entire functions, and Picard's great theorem about essential singularities.