Real Analysis Qualifying Exam
August 2011

Each problem is worth ten points. Work each problem on a separate piece of paper.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Give the definitions of convergence a.e. and convergence in measure for a sequence of measurable functions on X.

   (b) Show that every sequence of measurable functions on X which converges in measure to 0 has a subsequence which converges a.e. to 0.

2. Let X be a separable Banach space. Show that there exists an isometric linear map from X into ℓ^∞. Also, show that this is false in general if ℓ^∞ is replaced by ℓ^2.

3. Let X be a locally compact metric space and let $\{x_k\}_{k=1}^\infty$ be a sequence in X which has no convergent subsequence. Show that $\{n^{-1}\sum_{k=1}^n \delta_{x_k}\}_{n=1}^\infty$ converges to 0 in the weak* topology on $C_0(X)^*$, where δ_{x_k} denotes the point mass at x_k.

4. Let $\mathcal{P}$ be the set of all polynomials f on $[0,1]$ such that $f(0) = f'(0) = 0$. Determine, with proof, the values of p with $1 \leq p \leq \infty$ such that $\mathcal{P}$ is dense in $L^p[0,1]$.

5. Let $1 < p < \infty$, and let $\{x_k\}_{k=1}^\infty$ be a sequence in $\ell^p(\mathbb{N})$ such that $\lim_{k\to\infty} x_k(n) = 0$ for all $n \in \mathbb{N}$. Show that if there is an $M > 0$ such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$ then $x_k \to 0$ weakly. Also, show that if there is no such M then $\{x_k\}_{k=1}^\infty$ can fail to converge weakly.

6. Let $f \in C_0(\mathbb{R})$ and for every $t \in \mathbb{R}$ define $f_t \in C_0(\mathbb{R})$ by $f_t(x) = f(x+t)$ for all $x \in \mathbb{R}$.

   (a) Prove that $\{f_t : t \in [0,1]\}$ is compact in the norm topology.

   (b) Prove that $\{f_t : t \in \mathbb{R}\}$ is relatively compact in the weak topology.

7. Let f be an arbitrary real valued function on $[0,1]$. Show that the set of points at which f is continuous is a Lebesgue measurable set.

8. Show that not every nonempty bounded closed subset of ℓ^2 has a point of minimal norm, but that every nonempty bounded closed convex subset of ℓ^2 has a point of minimal norm.

9. Show that there is a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions on $[0,1]$ such that

   (a) $|f_n(t)| = 1$ for all n and all $t \in [0,1]$, and

   (b) for all $g \in L^1[0,1]$ one has $\int_0^1 f_n(t)g(t)dt \to 0$ as $n \to \infty$.

10. (a) Define what it means for a real valued function on $[0,1]$ to be absolutely continuous.

    (b) Prove that if f and g are absolutely continuous strictly positive functions on $[0,1]$ then f/g is absolutely continuous on [0,1].