Real Analysis Qualifying Exam; August, 2013.

Work as many of these ten problems as you can in four hours. Start each problem on a new sheet of paper.

#1. Let $1 \le p \le \infty$ and let $f \in L^p(\mathbf{R})$. For $t \in \mathbf{R}$, let $f_t(x) = f(x-t)$ and consider the mapping $G : \mathbf{R} \to L^p(\mathbf{R})$ given by $G(t) = f_t$. The space $L^p(\mathbf{R})$ is equipped with the usual norm topology.

(a) Show that G is continuous if $1 \le p < \infty$.

(b) Find an f for which the mapping G is not continuous when $p = \infty$ (and justify your answer).

(c) Let $1 \le p, q \le \infty$ be conjugate exponents (i.e., satisfying $\frac{1}{p} + \frac{1}{q} = 1$). Let $f \in L^p(\mathbf{R})$ and $g \in L^q(\mathbf{R})$ and show that their convolution $h = f * g$ is continuous. Recall

$$h(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,dx.$$

#2. (a) For $f \in C_{\mathbf{R}}([0,1])$, show that $f \ge 0$ if and only if $\|\lambda - f\|_u \le \lambda$ for all $\lambda \ge \|f\|_u$, where $\|\cdot\|_u$ denotes the uniform (supremum) norm.

(b) Suppose $E \subseteq C_{\mathbf{R}}([0,1])$ is a closed subspace containing the constant function 1. For $\phi \in E^*$, we define $\phi \ge 0$ to mean $\phi(f) \ge 0$ whenever $f \in E$ and $f \ge 0$. Show $\phi \ge 0$ if and only if $\|\phi\| = \phi(1)$.

(c) If $\phi \in E^*$ and $\phi \ge 0$, show that there is a bounded linear functional ψ on $C_{\mathbf{R}}([0,1])$ so that $\psi \ge 0$ and the restriction of ψ to E is ϕ.

#3. (a) Let μ and λ be mutually singular complex measures defined on the same measurable space $(X, \mathcal{M})$ and let $\nu = \mu + \lambda$. Show $|\nu| = |\mu| + |\lambda|$.

(b) Construct a nonzero, atomless Borel measure on $[0,1]$ that is mutually singular with respect to Lebesgue measure.

#4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions on $[0,1]$ and suppose that for all $x \in [0,1]$, $f_n(x)$ is eventually nonnegative. Show that there is an open interval $I \subseteq [0,1]$ such that for all n large enough, f_n is nonnegative everywhere on I.

#5. Let μ be a nonatomic *signed* measure on a measurable space (X, Ω), with $\mu(X) = 1$. Show that there is a measurable subset $E \subset X$ with $\mu(E) = 1/2$.

#6. Compute

$$\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx$$

and justify your computation.

#7. Prove or disprove: for every real-valued continuous function f on $[0,1]$ such that $f(0)=0$ and every $\epsilon>0$, there is a real polynomial p having only odd powers of x, i.e., p is of the form

$$p(x) = a_1x + a_3x^3 + a_5x^5 + \cdots + a_{2n+1}x^{2n+1},$$

such that $\sup_{x\in[0,1]}|f(x)-p(x)|<\epsilon$.

#8. Let $f\in L^1_{\text{loc}}(\mathbf{R})$. (a) What (by definition) are the Hardy–Littlewood maximal function Hf and the Lebesgue set L_f of f?

(b) State the Hardy–Littlewood Maximal Theorem.

(c) In each case, either construct concretely an example of f with the required property, or explain why no such example exists (you may use theorems from Folland about the Lebesgue set, if you state them).

(i) $L_f = \mathbf{R}$

(ii) the complement of L_f is uncountable

(iii) $L_f \subseteq (-\infty,0]\cup[1,\infty)$.

#9. Let X be a separable Banach space, let $\{x_n \mid n\geq 1\}$ be a countable, dense subset of the unit ball of X and let B be the closed unit ball in the dual Banach space X^* of X. For $\phi,\psi\in B$, let

$$d(\phi,\psi) = \sum_{n=1}^{\infty} 2^{-n}|\phi(x_n)-\psi(x_n)|.$$

Show that d is a metric on B whose topology agrees with the weak*-topology of X^* restricted to B.

#10. Let $T : X\to Y$ be a linear map between Banach spaces that is surjective and satisfies $\|Tx\|\geq\epsilon\|x\|$ for some $\epsilon>0$ and all $x\in X$. Show that T is bounded.