Real Analysis Qualifying Exam; January, 2009.

Work as many of these ten problems as you can in four hours. Start each problem on a new sheet of paper.

#1. Let $F \subset \mathbf{R}^n$ be compact and prove that the convex hull $\mathrm{conv}(F)$ is compact. (You may use without proof the theorem of Carathéodory that states that every point in the convex hull of any subset S of $\mathbf{R}^n$ is a convex combination of $n+1$ or fewer points of S.)

#2. Let $(X, \mathcal{M}, \rho)$ be a finite measure space. Suppose $\mathfrak{A} \subseteq \mathcal{M}$ is an algebra of sets and $\mu : \mathfrak{A} \to \mathbf{C}$ is a complex, finitely additive measure such that $|\mu(E)| \leq \rho(E) < \infty$ for all $E \in \mathfrak{A}$. Show that there is a complex measure $\nu : \mathcal{M} \to \mathbf{C}$, whose restriction to $\mathfrak{A}$ is μ, and such that $|\nu(E)| \leq \rho(E)$ for all $E \in \mathcal{M}$. (Hint: you may want to consider the set of simple functions of the form $\sum_1^n c_i 1_{E_i}$, for $E_i \in \mathfrak{A}$.)

#3. Given $1 \leq p \leq \infty$ and $f \in L^p([0,\infty))$, prove

$$\lim_{n\to\infty} \int_0^\infty f(x) e^{-nx}\, dx = 0.$$

#4. For each bounded, real–valued, Lebesgue measurable function f on $[0,1]$ prove that the sets

$$\begin{aligned} U(f) &= \{(x,y) \mid x \in [0,1],\ y \geq f(x)\}, \\ L(f) &= \{(x,y) \mid x \in [0,1],\ y \leq f(x)\}, \\ G(f) &= \{(x, f(x)) \mid x \in [0,1]\} \end{aligned}$$

are Lebesgue measurable subsets of $[0,1] \times \mathbf{R}$. (You may want to consider simple functions first.) Then prove that $G(f)$ is a null set (with respect to Lebesgue measure).

#5. Let $\phi : C_0(\mathbf{R}) \to \mathbf{C}$ be a bounded linear functional and suppose μ is a complex Borel measure on $\mathbf{R}$ such that $\phi(f) = \int f\, d\mu$ for every rational function f over the field of complex numbers whose restriction to $\mathbf{R}$ belongs to $C_0(\mathbf{R})$. Show that the formula $\phi(f) = \int f\, d\mu$ holds for all $f \in C_0(\mathbf{R})$.

#6. Let T be a surjective linear map from a Banach space X to a Banach space Y satisfying $\|Tx\| \geq \frac{1}{2009}\|x\|$ for all $x \in X$. Show that T is bounded.

#7. Let X be an infinite dimensional Banach space. Show
(a) the unit ball $\{x \in X \mid \|x\| \leq 1\}$ is closed in the weak topology on X,
(b) every nonempty, weakly open subset of X is unbounded and
(c) the weak topology on X is not the topology of a complete metric on X.

#8. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $E_n \in \mathcal{M}$ are such that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty. \tag{1}$$

Show

$$\mu(\limsup_{n\to\infty} E_n) = 0, \tag{2}$$

where $\limsup_{n\to\infty} E_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$.

(b) Either prove or disprove that the conclusion (2) follows when hypothesis (1) is replaced by

$$\sum_{n=1}^{\infty} \mu(E_n)^2 < \infty.$$

#9. Let $K : [0,1] \times [0,1] \to \mathbf{R}$ be continuous. If $f \in L^1([0,1])$, set

$$(Tf)(x) = \int_0^1 K(x,y)f(y)dy, \qquad (x \in [0,1]).$$

(a) Show $Tf \in C([0,1])$.
(b) Let B be the unit ball of $L^1([0,1])$ and show that $T(B)$ is relatively compact in $C([0,1])$.

#10. Let μ be a finite Borel measure on $\mathbf{R}$ that is absolutely continuous with respect to Lebesgue measure and show that for every Borel subset A of $\mathbf{R}$, the map $t \mapsto \mu(A+t)$ is continuous from $\mathbf{R}$ to $[0,\infty)$, where $A + t = \{s + t \mid s \in A\}$. (Hint: you might first suppose A is an interval.)