REAL ANALYSIS QUALIFYING EXAMINATION
JANUARY 2015

10 pts per question. Start each problem on a separate sheet. Results from Folland's book can be quoted without proof but should either be named or be carefully stated.

1. Let $f \in L^1(\mathbb{R})$. If
$$\int_a^b f(x)\,dx = 0$$
for all rational numbers $a < b$, prove that $f(x) = 0$ for almost all $x \in \mathbb{R}$.

2. Let $\{g_n\}_{n=1}^{\infty}$ and g be in $L^1(\mathbb{R})$ and satisfy
$$\lim_{n\to\infty} \|g_n - g\|_1 = 0.$$
Prove that there is a subsequence of $\{g_n\}_{n=1}^{\infty}$ that converges pointwise almost everywhere to g.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\mathcal{N} \subseteq \mathcal{M}$ be a σ-algebra. If $f \geq 0$ is $\mathcal{M}$-measurable and μ-integrable then prove that there exists an $\mathcal{N}$-measurable and μ-integrable function $g \geq 0$ so that
$$\int_E g\,d\mu = \int_E f\,d\mu, \qquad E \in \mathcal{N}.$$

4. (i) State the closed graph theorem.
   (ii) If H is a Hibert space and $T : H \to H$ is a linear operator satisfying
$$\langle Tx, y\rangle = \langle x, Ty\rangle, \qquad x, y \in H,$$
then prove that T is bounded.

5. Let $f, g \in L^1(\mathbb{R})$. Prove that $h \in L^1(\mathbb{R})$, where $h(x)$ is defined by
$$h(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy$$
whenever this integral is finite.

6. Let $f, g \in C[0,1]$ with $f(x) < g(x)$ for all $x \in [0,1]$.

   (i) Prove that there is a polynomial $p(x)$ so that
   $$f(x) < p(x) < g(x), \qquad x \in [0,1].$$

   (ii) Prove that there is an increasing sequence of polynomials $\{p_n(x)\}_{n=1}^{\infty}$ so that
   $$f(x) < p_n(x) < g(x), \qquad x \in [0,1],$$
   and $p_n \to g$ uniformly on $[0,1]$.

7. If $f \in L^2(\mathbb{R})$, $g \in L^3(\mathbb{R})$, and $h \in L^6(\mathbb{R})$ then prove that the product fgh is in $L^1(\mathbb{R})$.

8. (i) A point y in a metric space Y is isolated if the set $\{y\}$ is both open and closed in Y. Prove that $y \in Y$ is isolated if and only if the complement $\{y\}^c$ is not dense in Y.

   (ii) Let X be a countable nonempty complete metric space. Prove that the set of isolated points is dense in X.

9. Suppose that $f \in L^p(\mathbb{R})$ for all $p \in (1,2)$ and that $\sup_{p\in(1,2)} \|f\|_p < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that
   $$\lim_{p\to 2-} \|f\|_p = \|f\|_2.$$

10. Let $(X, \|\cdot\|)$ be a normed vector space with a subspace Y and let $\|\cdot\|_1$ be another norm on Y that satisfies
    $$\frac{1}{K}\|y\|_1 \le \|y\| \le K\|y\|_1, \quad y \in Y,$$
    where $K > 1$ is a fixed constant. Define S to be the set of linear functionals $\phi : X \to \mathbb{R}$ satisfying

    (a) $|\phi(y)| \le \|y\|_1, \qquad y \in Y,$

    (b) $|\phi(x)| \le K\|x\|, \qquad x \in X.$

    Prove the following statements:

    (i) $\|x\|_2 := \sup\{|\phi(x)| : \phi \in S\}$ defines a norm on X.

    (ii) For $y \in Y$, $\|y\|_1 = \|y\|_2$.

    (iii) The norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on X.