TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
August 2021

- There are 10 problems. Work on all of them and prove your assertions.
- Use a separate sheet for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Q1. Let X_i, $i \in I$ be topological spaces.

(a) Define the product topology on $\prod_{i\in I} X_i$.

(b) Let $\pi_i\colon \prod_{i\in I} X_i \to X_i$ be the projection. Prove that π_i is continuous with respect to both the product and the box topology.

(c) Suppose that Y is a topological space and

$$f\colon Y \to \prod_{i\in I} X_i$$

is a function. Prove that f is continuous with respect to the product topology if and only if for each $i \in I$, π_i is continuous.

(d) Suppose that $I = \mathbb{N}$ and $X_i = \mathbb{R}$ for each $i \in \mathbb{N}$. Consider the function

$$f\colon \mathbb{R} \to \prod_{i\in I} X_i$$

defined via $t \mapsto (t, t, t, ...)$. Is f continuous with respect to the product topology? Is f continuous with respect to the box topology? Explain or prove your answer.

Q2. Let X be a topological space.

(a) Let $\sim$ be an equivalence relation on X. Define the quotient topology on $X/\!\sim$.

(b) Let $g\colon Y \to X$ and $f\colon X \to Y$ be continuous functions such that $(f \circ g)(y) = y$ for all $y \in Y$. Prove that $U \subseteq Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open.

Q3. (a) A continuous function $f\colon X \to Y$ is called perfect if f is closed and the set $f^{-1}(y)$ is compact for each $y \in Y$. Prove that if $f\colon X \to Y$ is a perfect mapping onto Y , then $f^{-1}(Z)$ is compact for each compact $Z \subset Y$.

(b) Prove that if a topological space X is locally compact, Hausdorff, and second countable, then it is metrizable.

Q4. Let X be a topological space.

(a) Define "X is compact".

(b) Suppose that X is a Hausdorff space and that $A \subseteq X$ is a compact subspace. Prove that if $x \notin A$, then there exist disjoint open subsets of X that contain x and A respectively.

(c) Suppose that X is a compact Hausdorff space. Prove that X is regular.

Q5. Let X be a topological space.

(a) Define "X is normal".

(b) Let X be a connected normal space, which is also Hausdorff and which contains at least two points. Show that X is uncountable.

Q6. Consider the subset in $\mathbb{R}^2$

$$C := \{(x,y) \in \mathbb{R}^2 \mid y^2 = \frac{1}{3}x^3 + ax + b\}$$

where a, b are two real constants. Find out for what values of a and b the subset C is a smooth submanifold of $\mathbb{R}^2$. (Hint: use the implicit function theorem.)

Q7. Suppose a regular surface in $\mathbb{R}^3$ has first fundamental form

$$I = E(u,v)du^2 + G(u,v)dv^2$$

where $E(u,v)$ and $G(u,v)$ are smooth (positive-valued) function of u and v. Prove that the Gauss curvature is

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{\partial_v E}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{\partial_u G}{\sqrt{EG}}\right)\right].$$

Q8. Let M be a smooth manifold. Suppose $F : \Gamma(TM) \to C^\infty(M)$ is an $\mathbb{R}$-linear map[1] such that for all smooth functions $f \in C^\infty(M)$ and all smooth vector fields $X \in \Gamma(TM)$

$$F(fX) = fF(X).$$

(a) Prove the following statement: for a given $p \in M$, if $X_1 = X_2$ over some neighborhood U of p, then $F(X_1)(p) = F(X_2)(p)$.

(b) Prove that there exists a 1-form $\alpha \in \Omega^1(M)$ such that for all vector fields X,

$$F(X)(p) = \langle \alpha(p), X(p) \rangle_p, \ \forall p \in M.$$

Here $\langle \cdot, \cdot \rangle_p$ is the pairing between tangent and cotangent vectors at the point p.

Q9. Consider the complement of two points of the plane

$$M := \mathbb{R}^2 \setminus \{(1,0), (-1,0)\}.$$

For a given pair of real numbers (a_-, a_+), find a differential 1-form $fdx + gdy$ on M whose integral along the radius ϵ circle centered at $(\pm 1, 0)$ (oriented counterclockwise, $\epsilon > 0$ is an arbitrary small number) is $a_\pm$.

Q10. Let M be a smooth manifold. Suppose $\gamma : [0, +\infty) \to M$ is an integral curve of a smooth vector field X on M and suppose $\gamma(t)$ converges to a point $p \in M$ as $t \to \infty$. Prove that $X(p) = 0$.

[1] $\Gamma(TM)$ is the set of smooth vector fields on M and $C^\infty(M)$ is the set of smooth functions on M. They are both infinite-dimensional vector spaces over $\mathbb{R}$.