Topology/Geometry Qualifying Examination
January 2011

Answer all questions. Write your name and page number in the upper right corner of each page. Start each problem on a new sheet of paper, and use only one side of each sheet.

Notation. $\mathbb{R}$ denotes the real numbers, and $\mathbb{R}^n$ denotes Euclidean n-dimensional space. $\mathbb{S}^{n-1}$ is the unit sphere centered at the origin in $\mathbb{R}^n$.

1. Let X be a Hausdorff topological space.

(a) Show that every locally compact subspace of X is the intersection of two subsets of X, one of which is open and the other closed.

(b) Show that if X is locally compact, then X is completely regular.

2. A continuous function $f : X \to Y$ is called *perfect* if f is closed and the set $f^{-1}(y)$ is compact for each $y \in Y$. Prove that if $f : X \to Y$ is a perfect mapping onto Y, then $f^{-1}(Z)$ is compact for each compact $Z \subset Y$.

3. Let A be a connected subset of a connected space X, and $B \subset X - A$ be an open and closed set in the topology of the subspace $X - A$ of the space X. Prove that $A \cup B$ is connected.

4. If a collection $\mathcal{F}$ of subsets of a space X is locally finite and $\overline{A}$ is compact for each A in $\mathcal{F}$, then each $A \in \mathcal{F}$ intersects only a finite number of elements of $\mathcal{F}$.

5. Let M^n be a smooth n-dimensional manifold.

(a) State the definition of a smooth n-dimensional manifold. Define the tangent bundle of M.

(b) M^n is *parallelizable* if there exist n vector fields on M which are independent at each point of M. Prove that $\mathbb{S}^{n-1} \times \mathbb{R}$ is parallelizable for all n.

6. Let (X, ρ) be a metric space.

(a) Show that X is compact if and only if every sequence in X has a convergent subsequence.

(b) Assume that X is compact. Let $f : X \to X$ be an *isometry*; that is, $\rho(x, y) = \rho(f(x), f(y))$ for all $x, y \in X$. Prove that f is a mapping onto X.

7. Give an example of an immersion $f : \mathbb{R} \to \mathbb{R}^2$ that is not an embedding. (Full justification is required: prove that your example is an immersion, but not an embedding.)

8. Consider the smooth map $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x, y, z) = \left((x-y)^2, (x-y)(y-z), (y-z)^2\right).$$

(a) What is the maximum rank that F achieves?

(b) At which points $(x, y, z) \in \mathbb{R}^3$ is the rank of F less than the maximum?

9. Assume that all functions are maps of Euclidean spaces.

(a) State the Inverse Function Theorem.

(b) State the Implicit Function Theorem.

(c) Assume that the Inverse Function Theorem holds, and prove the Implicit Function Theorem.

10. Let T^2 denote the two-dimensional torus with the standard structure as a differentiable manifold.

(a) Prove T^2 admit a flat metric; that is, a metric with Gauss curvature identially zero.

(b) Use part (a) and the Gauss-Bonnet Theorem to complete the Euler characteristic of T^2.

(c) Does T^2 admit a metric that is not flat with Gauss curvature $K \geq 0$? (Justify your answer.)

11. Prove Cartan's Lemma: Let M be a smooth manifold of dimension n. Fix $1 \leq k \leq n$. Let ω^i and ϕ_i be 1-forms on M. Suppose that the $\{\omega^1, \ldots, \omega^k\}$ are point-wise linearly independent, and that $0 = \sum_{i=1}^k \phi_i \wedge \omega^i$. Prove that there exist smooth functions $h_{ij} = h_{ji} : M \to \mathbb{R}$ such that for all $i = 1, \ldots k$, $\phi_i = \sum_{j=1}^k h_{ij}\, \omega^j$.

12. Prove that a Lie group admits a global framing.

