TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
January 2021

- `There are 10 problems. Work on all of them and prove your assertions.`
- `Use a separate sheet for each problem and write only on one side of the paper.`
- `Write your name on the top right corner of each page.`

Q1. Define a topology on the set $\mathbb{R}$ of real numbers by the condition that $U \subseteq \mathbb{R}$ is open if and only if it is either empty or contains the interval $[0, 1)$. Then

(a) What is the interior of the set $[0, 1]$? And its closure? Prove your answer.
(b) Does this topology on $\mathbb{R}$ satisfy the T_0 condition? Explain your answer.
(c) Is $\mathbb{R}$ connected in this topology? Prove your answer.
(d) Is $\mathbb{R}$ compact in this topology? Prove your answer.

Q2. Let X be a topological space, and let Y be any set. A function $f\colon X \to Y$ is called locally constant if for every $x \in X$, there is an open set $U_x \subseteq X$ containing x such that f is constant on U_x. Prove that a locally constant function on a connected space is constant.

Q3. (a) State the Urysohn Metrization Theorem. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X is second countable.
(b) Show that the uncountable product of intervals $[0, 1]$ is not first countable, and therefore is not metrizable.

Q4. Let X and Y be two locally compact Hausdorff topological spaces.

(a) Define the one-point compactification of X. Make sure to define it as a topological space, i.e. specify the topology on it.
(b) A map is called proper if the inverse image of any compact set is compact. Prove that a continuous map $f\colon X \to Y$ is proper if and only if it extends to a continuous map between the one-point compactifications of X and Y.

Q5. Let X be compact and Hausdorff. Let $f\colon X \to Y$ be continuous, closed, and surjective. Prove that Y is Hausdorff.

Q6. (a) Formulate the Implicit Function theorem;
(b) Let M be a smooth manifold. Two embedded submanifolds $S_1, S_2 \subset M$ are said to be *transverse* if for esch $p \in S_1 \cap S_2$, the tangent spaces T_pS_1 and T_pS_2 together span T_pM, i.e.

$$T_pM = T_pS_1 + T_pS_2.$$

If S_1 adn S_2 are transverse, show that $S_1 \cap S_2$ is an embedded submanifold of M of dimension $\dim S_1 + \dim S_2 - \dim M$.
(c) Give a counterexample to the conclusion of the previous item, if S_1 amd S_2 are not transverse.

Q7. Let M_1 and M_2 be two smooth manifolds and $\varphi\colon M \to N$ a smooth map.

(a) Write down the definition for the map $\varphi^*\colon \Omega^k(M_2) \to \Omega^k(M_1)$ that pulls k-forms on M_2 to k-forms on M_1.

(b) Show that

$$\varphi^*(d\omega) = d(\varphi^*\omega)$$

for all k-forms ω on M_2.

Q8. Let M be a smooth manifold and X a smooth vector field on M.

(a) Prove that if $X(p) \neq 0$ for some $p \in M$, then there is a coordinate system (x, U) around p such that

$$X = \frac{\partial}{\partial x_1} \text{ on } U.$$

(b) If Y is another smooth vector field on M such that $[X, Y] = 0$ on M, then show that

$$\varphi_s \circ \psi_t = \psi_t \circ \varphi_s$$

for all $s, t \in \mathbb{R}$, where $\{\varphi_t\}$ (resp. $\{\psi_s\}$) is the 1-parameter group of diffeomorphisms generated by X (resp. Y).

Q9. Show that if the Riemannian metric g in local coordinates (x, y) on a 2-dimensional manifold is given by

$$g = dx^2 + 2\cos\omega(x,y)dx\,dy + dy^2$$

for some function ω, then the Gaussian curvature

$$K = -\frac{\omega_{xy}}{\sin\omega}.$$

Q10. Suppose M and N are two closed oriented smooth manifolds of the same dimension and $\varphi\colon M \to N$ is a smooth map.

(a) Let ω be an n-form on N such that $\int_N \omega \neq 0$. We define the degree of the map φ to be the number a such that

$$\int_M \varphi^*\omega = a \cdot \int_N \omega.$$

Show that the degree of φ is independent of the choice of ω.

(b) Show that there exists a smooth map of degree 1 from any closed oriented smooth n-dimensional manifold to S^n.