Texas A&M Mathematics Department Mini Fair Math Contest

Grades 5-7 Solutions

April 16, 2011

1. $27 \times 53 = 1431$

2. $41 \div 5 = 8.2$ or $8\frac{1}{5}$

3. A single pad of paper is .5 inches thick and weighs 3 ounces. What would be the weight of a stack of such pads that is 12 feet high?

Solution. 1 ft consists of 24 pads, so 12 feet consists of $12 \times 24 = 288$ pads. This stack would weigh $288 \times 3 = 864$ ounces, which is 54 lbs.

4. At the end of the year, my salary was three times its size at the beginning of the year. What was the percent increase of my salary from the beginning to the end of the year?

Solution. If the initial salary is x then the end salary is $3x$ and therefore the increase is $2x$ which is 200 percent increase.

5. Find the value of

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{5}}.$$

Your answer should be a single fraction in reduced form.

Solution. Compute

$$\frac{\frac{3}{6}+\frac{2}{6}}{\frac{5}{20}+\frac{8}{20}} = \frac{\frac{5}{6}}{\frac{13}{20}} = \frac{50}{39}.$$

6. Determine which shape in the figure below has the largest area.

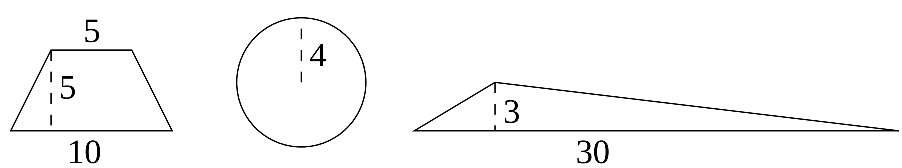

FIGURE 1. Figure for Problem 6

Solution. The area of the trapezoid is

$$A_{\text{trap}} = \frac{1}{2}(a+b)h = \frac{1}{2} \times 15 \times 5 = \frac{75}{2}.$$

The area of the circle is

$$A_{\text{circle}} = \pi \times r^2 = \pi \times 16 > 48.$$

The are of the triangle is

$$A_{\text{triangle}} = \frac{1}{2}bh = \frac{1}{2} \times 30 \times 3 = \frac{90}{2}.$$

7. Which statement is ALWAYS true?

a. The diagonals of a parallelogram are perpendicular.
b. The diagonals of a parallelogram bisect each other.
c. The diagonals of a parallelogram are congruent.
d. Adjacent sides of a parallelogram are congruent.
e. Adjacent angles of a parallelogram are congruent.

Solution. It is easy to see that **b.** is always true and the rest could be false.

8. Luke made $\frac{2}{3}$ of a gallon of hot chocolate. If cups are $\frac{1}{10}$ of a gallon, how many cups can he fill, and how much will be left over? Your answer should be a number of cups and an amount left over in gallons.

Solution. He can make

$$\frac{\frac{2}{3}}{\frac{1}{10}} = \frac{20}{3} = 6\,\frac{2}{3}.$$

So 6 cups and $\frac{2}{3} \times \frac{1}{10} = \frac{1}{15}$ gallons.

9. What is the 25th digit in the decimal representation of $1/13$?

Solution. The fraction $1/13$ has the periodic representation $0.(076923)$. Therefore the 25th digit to the right of the decimal is 0.

10. Determine the next two numbers in the pattern

$$31, 62, 93, 25, 56, 87, \ldots$$

Solution. The pattern is add 31 twice, subtract 68, add 31 twice etc. Subtracting 68 from 87 we obtain 19, and adding 31 we get 50.

11. There are three tribes on an island. Members of the Liar Tribe always lie. Members of the Truthful Tribe always tell the truth. Members of the Alternator Tribe always alternate their statements: they tell one lie, then one true statement, then one lie, and so on. Which situation is impossible?

a. A Liar says, I am a Liar.
b. A Truthful says, I am a Truthful.
c. An Alternator says, I am a Liar.
d. A Liar says, I am a Truthful.
e. An Alternator says, I am an Alternator.

Solution. A Liar cannot tell the true which makes **a.** impossible.

12. Find the smallest integer n so that

$$1 + 2 + 3 + \cdots + n > 100.$$

Solution. Either simply start adding the numbers until you get there, or use

$$\frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n$$

to conclude $n = 14$.

13. A large cube is made up of identical unit cubes. After the unit cubes are glued together to form the large cube, it is dipped in paint. For example, 27 unit cubes could be assembled into a 3x3x3 cube. After it was dipped in paint, 8 of the unit cubes would have three painted faces, 12 would have two painted faces, 6 would have one painted face, and 1 would have no painted faces. If 1,728 unit cubes were assembled into an 12x12x12 cube and then dipped into paint, how many of the unit cubes would then have three painted faces?

Solution. In order to have three painted faces a unit cube must be in one of the 8 corners (just like in the 3x3x3 cube case). Therefore, there are 8 unit cubes with three painted faces.

14. What is the sum of the first 100 even numbers.
Using that the sum of the first 100 even numbers is twice the sum of the first 100 numbers, and the formula

$$\frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n,$$

we conclude that the sum is 10100.

15. A farmer has several black and several white horses. If one of the black horses were white, the number of black horses would be equal to the number of white ones. If one of the white horses were black, the number of black horses would be twice as large as the number of white ones. How many black horses and how many white horses does the farmer have?

Solution. Let b denote the number of black horses and w the number of white horses, so that the conditions respectively give the system

$$\begin{aligned} b - 1 &= w + 1 \\ b + 1 &= 2(w - 1). \end{aligned}$$

Subtracting the second from the first, we find $-2 = -w + 3$, so that $w = 5$ and $b = 7$.

16. Which of the following is impossible?

a. A rectangle with area 25 square meters and perimeter 29 meters
b. A rectangle with area 25 square meters and perimeter 20 meters
c. A rectangle with area 16 square meters and perimeter 16 meters
d. A rectangle with area 16 square meters and perimeter 20 meters
e. A rectangle with area 9 square meters and perimeter 9 meters

Solution. If a and b are the sides of the rectangle then the area is ab and the perimeter is $2(a+b)$. We have

$$0 \leq (a-b)^2 = (a+b)^2 - 4ab$$

which implies $2(a+b) \geq 4\sqrt{ab}$ and therefore **e.** is impossible ($9 \geq 4\sqrt{9}$).

17. In a group of five friends, the sums of the ages of each group of four of them are 124, 128, 130, 136, and 142. What is the age of the youngest of the friends?

Solution. Since each age is omitted from one of the five sums, we know

$$4 \times (x_1 + x_2 + x_3 + x_4 + x_5) = (124 + 128 + 130 + 136 + 142) = 660,$$

where x_1 denotes the age of the youngest, x_2 the age of the second youngest etc. The sum of all ages is $660/4 = 165$, and since we must have

$$x_2 + x_3 + x_4 + x_5 = 142,$$

we can write

$$x_1 = 165 - 142 = 23.$$

18. It is 1:00 a.m. How much money would you make in the next 48 hours if you made 8 dollars every time the hands of a clock formed a 90 degree angle?

Solution. During most one-hour periods the hands form a right angle twice, but not between 3 and 4 and not between 9 and 10. (I'm considering the times 3 and 9 o'clock to be counted in the prior periods.) This means that during a 12 hour period there are 22 right angles, and so over a 48 hour period $22 \times 4 = 88$. Since you make 8 dollars each time, you make $88 \times 8 = \$704$ total.

19. Which of the following is the smallest number divisible by 9 different prime numbers?

a. 111,500,000
b. 203,693,490,000
c. 223,092,870
d. 3,011
e. 3,234

Solution. It is easy to see that a,d, and e are not divisible by 9 different primes. The next smallest number is $223,092,870 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23$.

20. Evaluate $\sqrt[8]{\left(7\left(\frac{\sqrt{4.41}}{7}\right)\right)^8}$. (Round to the nearest hundredth.)

Solution. It is easy to see that

$$\sqrt[8]{\left(7\left(\frac{\sqrt{4.41}}{7}\right)\right)^8} = 7\left(\frac{\sqrt{4.41}}{7}\right) = \sqrt{4.41} = \sqrt{(2.1)^2} = 2.1$$

21. Allan and Bill are walking in the same direction beside a railroad track, and Allan is far behind Bill. Both walk at constant speeds, and Allan walks faster than Bill. A long train traveling at a constant speed in the same direction will take 10 seconds to pass Allan (from the front to the end) and will take 9 seconds to pass Bill. If it will take twenty minutes for the front of the train to travel from Allan to Bill, how many minutes will it take for Allan to catch up to Bill?

Solution. Denote the velocities of the train, Allan, and Bill respectively by V_T, V_A, and V_B, and let L denote the length of the train. Since it takes the train 10 seconds to pass Allan, we know

$$V_T = V_A + L/10,$$

while since it takes the train 9 seconds to pass Bill we know

$$V_T = V_B + L/9.$$

Finally, since it takes the train 1200 seconds (20 minutes) to get from Allan to Bill, we know

$$(V_T - V_B)1200 = d = \text{original distance between Allan and Bill.}$$

Our goal is to find the time T so that

$$(V_A - V_B)T = d.$$

Using the first two equations, we find

$$V_A - V_B = \frac{L}{90},$$

while the second and third equations give

$$d = \frac{1200L}{9}.$$

We conclude

$$\frac{L}{90}T = \frac{1200L}{9} \Rightarrow T = 12,000\text{seconds},$$

which is 200 minutes.

22. Which statement is true about $\left(\frac{10}{11}\right)^{111} \cdot \left(\frac{11}{10}\right)^{211}$?

a. The product is greater than 1,000.
b. The product is greater than 700 but less than 1000.
c. The product is greater than 3 but less than 700.
d. The product is greater than 1 but less than 3.
e. The product is less than 1 but greater than 0.

Solution. We have

$$\left(\frac{10}{11}\right)^{111} \cdot \left(\frac{11}{10}\right)^{211} = (1.1)^{100} = (1.21)^{50} = \left((1.21)^5\right)^{10}.$$

Using that $(1.21)^5 > (1.2)^4 > 2$, we conclude

$$\left(\frac{10}{11}\right)^{111} \cdot \left(\frac{11}{10}\right)^{211} = (1.1)^{100} = \left((1.21)^5\right)^{10} > 2^{10} = 1024 > 1000.$$