On bounds on n-variate $(n+3)$-nomial hypersurfaces

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August 18, 2011

Abstract

We investigate the bounds on the number of connected components of polynomial hypersurfaces, specifically those defined by n-variate $(n+3)$-nomials. We discuss the current methods used to prove these bounds and make a conjecture about new bounds and how they might be achieved.

1 Introduction

Descartes' rule of signs gives us a very simple rule for bounding the number of real positive roots of a univariate polynomial by arranging the terms in decreasing degree and counting sign changes. Such a bound is useful for numerically approximating solutions to polynomials, intuitively because, if the rule gives us a bound of at most r distinct roots, and we find r roots, we know that we need not search for any more. Bezout's theorem provides another nice bound for univariate polynomials, and is similar to Descartes' in nature, but bounds the number of solutions in a particular *interval* of the real line.

Of course, it would also be nice to have bounds of this nature for multivariate polynomials. Unfortunately, having multiple variables typically leads to infinite zero sets, and therefore we have to look for some bound other than on the number of roots. Recent bounds focus on the number of *connected components* of the positive real zero set, that is, distinct continuous curves in the zero set. There is also a distinction between *compact* and *noncompact* components; the former are closed and bounded, which intuitively correspond to "closed loops". Our research focuses on the compact components.

Currently, the best general bounds are not believed to be tight (that is, though we have bounds, we do not believe there are any polynomials that achieve it). Bihan and Sottile [2] give a bound of $\lfloor \frac{5n+1}{2} \rfloor$ for n-variate $(n+3)$-nomials, which currently is the best known bound, though we believe it is not tight. We suspect that a lower bound can be achieved. We look to the methods of Perrucci [4], which he used to characterize the zero sets of n-variate 4-nomials. We believe these methods could be used to characterize both n-variate $(n+2)$-nomials and n-variate $(n+3)$-nomials.

2 Preliminaries

Let us first clear up some notation for our discussion.

Notation. *Given* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and* $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, *we write*

$$\mathbf{x}^a = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{R}$$

The most important feature of this notation is that it allows us to view the exponents of monomial terms as vectors, which is almost necessary for understanding some results.

Just to make things clear, let us also define our notion of a polynomial:

Definition. *An n-**variate** m-**nomial** is a function* $f : \mathbb{R}^n \to \mathbb{R}$ *of the form*

$$f = \sum_{i=1}^{m} c_i \mathbf{x}^{a_i}$$

where $c_i \in \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, *and* $a_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{R}^n$.

We have chosen to focus on n-variate $(n+3)$-nomials.

Definition. *The* ***positive real zero set*** *of a function* $f : \mathbb{R}^n \to \mathbb{R}$ *is the set* $Z_+(f) := \{\mathbf{x} \in \mathbb{R}^n_+ : f(\mathbf{x}) = 0\}$.

We also make use of the following parameterization. It is similar to a parameterization used in [4]:

Definition. *Let* $\mathbf{p} = (p_1, \dots, p_n), \mathbf{q} = (q_1, \dots, q_n)$ *be fixed points in* $\mathbb{R}^n+$ *such that* $p_i \neq q_i$. *Define, for* $2 \leq i \leq n$,

$$v_i = \frac{\log(q_i/p_i)}{\log(p_1/q_1)}$$

The function $h_{(\mathbf{p},\mathbf{q})} : \mathbb{R} \to \mathbb{R}^n$ *is the function given by*

$$h_{(\mathbf{p},\mathbf{q})}(t) = (h^1_{(\mathbf{p},\mathbf{q})}, \dots, h^i_{(\mathbf{p},\mathbf{q})}(t)) = \begin{cases} t, & i = 1 \\ (p_1^{v_i} p_i) t^{-v_i}, & 2 \leq i \leq n \end{cases}$$

Furthermore, for any function $f : \mathbb{R}^2 \to \mathbb{R}$, *we define*

$$f_{(\mathbf{p},\mathbf{q})}(t) = (f \circ h_{(\mathbf{p},\mathbf{q})})(t)$$

Remark. *Note that* $p_1^v p_i = q_1^v q_i$:

$$\begin{aligned} p_1^{v_i} p_i &= \left(q_1 \frac{p_1}{q_1}\right)^{v_i} p_i \\ &= q_1^{v_i} \left(\frac{q_i}{p_i}\right) p_i \\ &= q_1^{v_i} q_i \end{aligned}$$

Furthermore, $h_{(p,q)}(t)$ *passes through both points* $\mathbf{p}$ *and* $\mathbf{q}$. *This is easy to observe by noting that the parameterization has a term* $p_1^v p_i$ *(which is also* $q_1^v q_i$*) that is multiplied by* $p_1^{-v_i}$ *when* $t = p_1$ *(alternatively* $q_1^{-v_i}$ *when* $t = q_1$*).*

3 Previous Results

As stated earlier, the best currently known bound for n-variate $(n+3)$-nomials is given in [2]:

Theorem 1. *Suppose* f *is an* n*-variate* $(n+3)$*-nomial. Then there is at most* $\lfloor \frac{5n+1}{2} \rfloor$ *compact connected components in* $Z_+(f)$.

The proof of this result relies on the Khovanskii-Rolle theorem and the use of Gale dual systems. Both are covered in the article.

There is also a general result given for n-variate $(n+2)$-nomials by Bihan, Rojas, and Stella [1]. These authors also provide bounds on the number of noncompact components for the same polynomials, although these are not our present concern.

Theorem 2. *Suppose* f *is an* n*-variate polynomial with no more than* $n+2$ *monomial terms. Then there is at most one compact connected component in* $Z_+(f)$.

Perrucci [4] affirms the bounds given in [1] for the case $n = 2$. Although he does not approach the general case in this article, his methods are of interest to us.

Finally, we make extensive use of the following "canonical form"; given any n-variate $(n+3)$-nomial, we can find a particular n-variate $(n+3)$-nomial that has this particular form.

Theorem 3. *Let* f *be an* n*-variate* $(n+3)$*-nomial. Then there exists a* g *of the form*

$$g = \pm 1 \pm x_1 \pm \dots \pm x_n + A\mathbf{x}^a + B\mathbf{x}^b,$$

where $a, b \in \mathbb{R}^n$ *and* $A, B \in \mathbb{R}^*$, *such that* $Z_+(f) = Z_+(g)$.

The proof of the above is demonstrated in [3, lemma 1]. The article does not explicitly discuss the method for n-variate $(n+3)$-nomials, but the same general method applies to yield this result.

4 Results

All of our work here is in effort to demonstrate that a 2-variate 5-nomial cannot have more than one compact connected component in its positive real zero set. For this reason, many of our results assume that such a polynomial does exist, for the purposes of contradiction. We have generalized where possible.

Our first result is elementary and concerns the relative positioning of such components:

Theorem 4. *Let f be an n-variate $(n+3)$-nomial with two compact connected components in $Z_+(f)$. Then there is no line parallel to any axis passing through the interiors of any two such components.*

Proof. Fix a variable $z = x_j$ for some j. We may assume using theorem (3) that f is of the form $(\pm 1 \pm x_1 \pm \cdots \pm x_{j-1} \pm x_{j+1} \pm \cdots \pm x_n) \pm x_j + A\mathbf{x}^a + B\mathbf{x}^b$. Now suppose there is a line parallel to an axis passing through the interiors of two compact connected components. This line can be parameterized by fixing the values of x_i for $i \neq j$ and allowing x_j to vary. But then the equation above is a 4-nomial in x_j, and therefore has at most three real positive roots. But since the line passes through the interiors of both compact connected components, there are at least four values for x_j where the polynomial is zero. By contradiction, there can exist no such line. $\square$

The rather important use of this theorem is that we know that there is no line passing through the critical points of a polynomial contained within the compact connected components.

A similar idea follows.

Theorem 5. *Let g be an n-variate $(n+3)$-nomial such that $Z_+(g)$ has two compact connected components.. Then there cannot be a line through the origin passing through two compact connected components' interiors.*

Proof. The proof is practically the same as above, except that we can parameterize a line through the origin in terms of linear multiples of x_j, so that we have $f = \pm 1 + (c_1 + \cdots + c_{j-1} \pm 1 + c_{j+1} + \cdots + c_n)x_j + A\mathbf{x}^a + B\mathbf{x}^b$. Note that once again this is a 4-nomial in x_j. $\square$

The following theorem tells us something more about 2-variate 5-nomials in particular. It borrows heavily from a method in [4]:

Lemma 1. *Suppose f is an 2-variate 5-nomial with two compact connected components in $Z_+(f)$. Then, without loss of generality, f has form*

$$f = \pm 1 - x - y + A\mathbf{x}^a + B\mathbf{x}^b$$

where $A, B > 0$.

Proof. We already know from theorem 3 that f has form $\pm 1 \pm x_1 \pm \cdots \pm x_n + A\mathbf{x}^a + B\mathbf{x}^b$. Since f is a polynomial function, and therefore everywhere differentiable, we know that f has at least two critical points, one contained within the interior of each compact connected component. Let $\mathbf{p}$ and $\mathbf{q}$ be two such points in distinct compact connected compnents. Then observe that $f_{(\mathbf{p},\mathbf{q})}(t)$ is a five-nomial with at least four zeros, and therefore must have at least four sign changes. But then we may assume that f has three positive and two negative terms, since we may multiply by -1 without altering $Z_+(f)$, for to have four sign changes, f must have four sign alterations, and therefore three terms of one sign and two terms of another. Furthermore, since we may choose which terms are positive and which are negative, when converting f to the canonical form, we may assume that A and B are positive, as is 1. $\square$

Remark. *We believe that if a lower bound for the number of compact components were to be found for 2-variate 5-nomials, it would be greatly assisted by the above form.*

We conclude with an alternate proof of the bound given by Bihan & Sottile.

Proof. As in the proof of 1, since f is everywhere differentiable, we know that it must have at least one critical point in the interior of every compact connected component. We know further that at each of these points all partial derivatives of f vanish. Observe, then, that the system of equations

$$x\frac{\partial f}{\partial x} = y\frac{\partial f}{\partial y} = 0$$

must hold at every such point. but we know that

$$x\frac{\partial f}{\partial x} = -x + a_1 A\mathbf{x}^a + b_1 B\mathbf{x}^b$$
$$y\frac{\partial f}{\partial y} = -y + a_2 A\mathbf{x}^a + b_1 B\mathbf{x}^b$$

and this is a 2×2 trinomial system, which was shown in [3, theorem 1] to have at most 5 roots. Thus f has at most five compact connected components. $\square$

References

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