Bounding the Number of Components of Positive Zero Sets

REU on Algorithmic Algebraic Geometry

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Review

Let $f(x) = c_0 + c_1 x^{a_1} + \cdots + c_n x^{a_n}$ where $c_i \in \mathbb{R}$, $a_i \in \mathbb{R}^n \, \forall i \in \mathbb{N}$.

Recall:
$Z_+(f) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ | f(x_1, \ldots, x_n) = 0\}$.
Similarly, $Z_{\mathbb{R}}(f) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | f(x_1, \ldots, x_n) = 0\}$.

Descartes' Rule:

1. for positive roots, start with the sign of the coefficient of the lowest power.
2. count the number of sign changes n as you proceed from the lowest to the highest power
3. then n is the maximum number of positive roots.

e.g.,

$$f(x) = 3 - 9x + 5x^3 + x^7$$

Statement

Proposition:

Given $f \in \mathbb{R}[x_1, \ldots, x_n]$ an honest $(n+2)$-nomial, $Z_+(f)$ has at most two connected components.

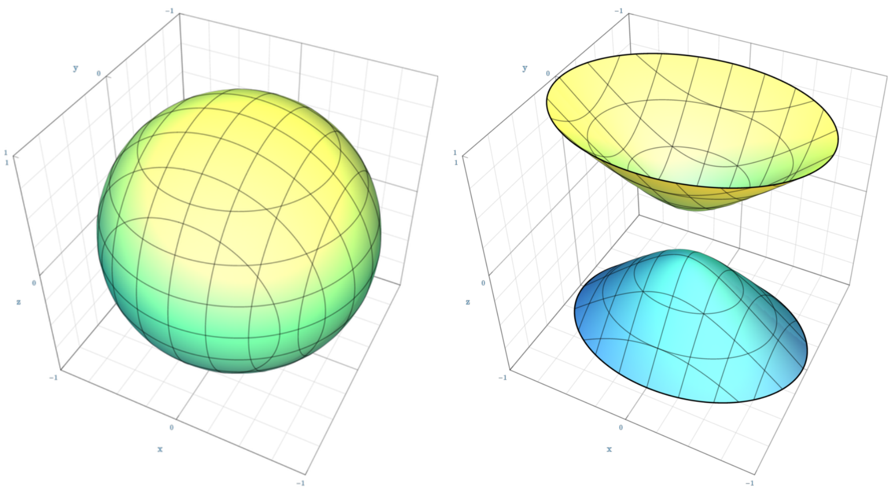

Figure 1: 1 and 2 connected components

Images courtesy of Wikipedia

Sketch of Proof

We proceed by induction. **Base Case:** Let f be an honest, n-variate $(n+2)$-nomial.
If $n = 1$ we have a univariate trinomial, which can have at most 2 sign changes and thus, by Descarte's Rule, at most 2 positive roots (i.e connected components).

$$f(x) = c_0 \pm c_1 x \pm c_2 x^2$$

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$$f_1(x_1, \ldots, x_n) = c_0 x^{a_0} + c_1 x^{a_1} + \ldots + c_{n+1} x^{a_{n+1}}$$

(We may assume that each $a_{ij} \geq 0$ by multiplying by the appropriate monomial, which leaves the positive zero set unaffected).

Simplification: We obtain an exponential sum f_2, where

$$f_2(x_1, \ldots, x_n) = 1 \pm e^{b_1 x} \pm \ldots \pm e^{b_n x} \pm \gamma e^{b_{n+1} x}$$

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via rescalings and changes of variables. If we then use a linear change of variables to obtain

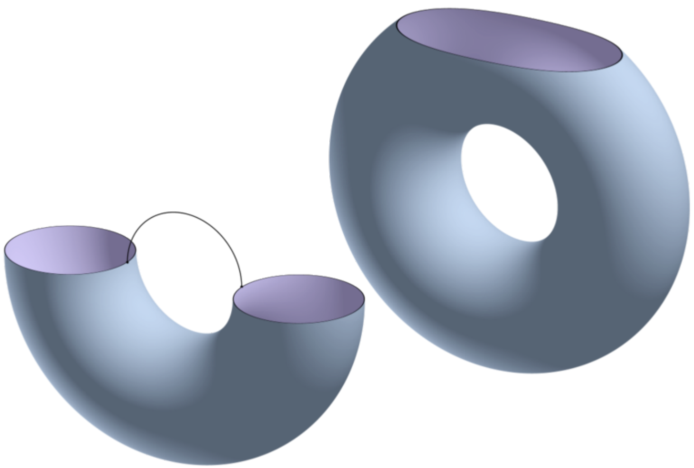
$$f_3(x_1, \ldots, x_n) = 1 \pm e^{x_1} \pm \ldots \pm e^{x_n} \pm k e^{b \cdot x}$$

(where $k \in \mathbb{R}_{>0}$), then f_3 has a zero set topologically equivalent to that of f_1.
Note that rewriting f_1 as an exponential function implies that we will examine $Z_{\mathbb{R}}(f_3)$ instead of $Z_+(f_1)$.

Morse Theory

Introduction

Let h be a smooth function $h : M \to \mathbb{R}$ with no degenerate critical points. **Morse Theory** provides a method of examining the topology of a manifold M (in our case, $Z_{\mathbb{R}}(f)$, $Z_{+}(f)$) using the behavior of h on M. By looking at the level sets of a space, we can gain insight to the topology of the whole space.



Definitions

- A **critical point** of a function is a root of all of the function's partial derivatives.
- A function h such that all of its critical points are nondegenerate is called a **Morse function**.
- A **critical value** is h evaluated at k.

Sketch of Proof Continued

Finding Critical Points: Now consider M to be the real zero set of f_3. Let $h(x_1, \ldots, x_{n-1}) = x_n$. The critical points of h on M must also satisfy the following system of equations:

$$\mathbf{H} = \begin{cases} \pm e^{x_1} = \pm\gamma\alpha_1 e^{\alpha x} \\ \vdots \\ \pm e^{x_{n-1}} = \pm\gamma\alpha_{n-1} e^{\alpha x} \end{cases}$$

Solving H

We have two cases:

Case one, the system **H** has no solutions.

Case two, the system **H** has at least one solution.

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- By our induction hypothesis, M_0 has at most 2 connected components.
- Thus, every diffeomorphic level set of M has at most 2 connected components, so M has at most 2 connected components.

With solutions

Case two: If there are solutions to the system of equations

$$\mathbf{H} = \begin{cases} \pm e^{x_1} = \pm\gamma\alpha_1 e^{\alpha x} \\ \vdots \\ \pm e^{x_{n-1}} = \pm\gamma\alpha_{n-1} e^{\alpha x} \end{cases}$$

We substitute into the defining function of M to obtain

$$1 \pm e^{x_n} \pm \gamma' e^{\alpha_n x_n}$$

By Descarte's Rule, there can be at most two solutions, i.e., at most 2 critical points.

One and Two Solutions

If there are solutions, say $k_1, k_2 \in M$, then k_1 and k_2 are critical points. We want to show that a path can be written between any point $m \in M$ and one of k_1 or k_2.

By proving case one and case two, we have shown that we will have 0, 1, or 2 critical points and, by induction and the application of Morse Theory, M will have at most two connected components. Thus, $Z_+(f)$ has at most 2 connected components.

Future Directions

Understanding connectedness and number of components are key parts in understanding the topology of $Z + (f)$. As we develop our algorithm, we will use this fact to understand $Z_+(f)$ and to gain intuition as to which quadric hypersurface a given positive real zero set may yield.

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