Polynomial System Solving Using Archimedean Tropical Varieties

Yasmina Marden and Hailey Stirneman

Summer 2015

Outline

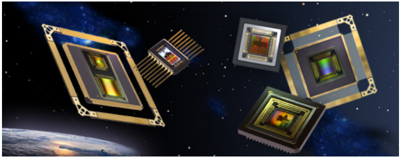
- Relevance of fast algorithms in practice.
- Position Estimation via Polynomial System Solving.
- Some Tropical Techniques.

Radiation

- Computers on spacecraft have to be made with circuits resistant to radiation.

Definiton

Radiation hardening means making electronic components and systems resistant to damage or malfunctions caused by ionizing radiation.



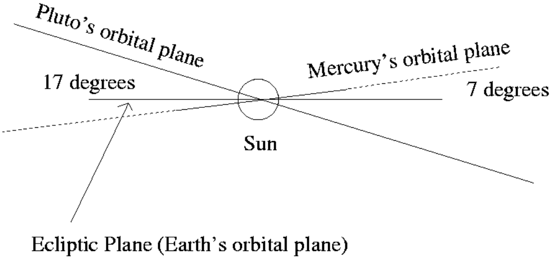
Importance of radiation hardening

- Cosmic rays cannot easily flip bits and cause errors.
- They continue to do accurate calculations when ordinary chips might glitch.

Price of radiation hardening

- The process is expensive, a radiation hardened single board computer can be around $200,000.
- Radiation hardened chips are power hungry.
- They can be up to 10x slower than an equivalent terrestrial CPU.

Suppose you are in space and you know the position of three planets. Also suppose further that there is no plane containing you and all the planets (which in fact happens with high probability), and that you can measure the angle between any two planets from your position.



So, how do we find your position?

We show how estimating your position can be reduced to polynomial system solving.

Equi-angular properies of chords and circles

The locus, Q, of which $\angle$AQB $= \theta$ forms a circle

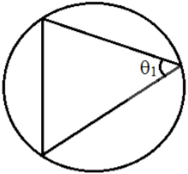
Lemma 1

Through any 3 non-collinear points, there is a unique circle.

Lemma 2

for any r,s with r, s $\neq$ -1 we have,
arctan(r) - arctan(s) $=$ arctan $\left(\frac{r-s}{1+rs}\right)$

This proof allows us to generate circles from which we get the polynomials in our system of equations that we will solve in order to find our location. We solve this by finding the intersection point of three semi-algebraic arcs.

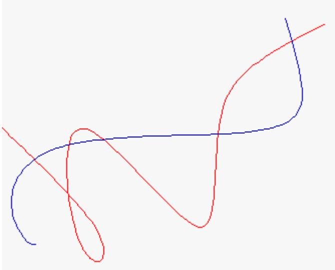


Condition numbers

- Condition numbers measure the sensitivity of a function to changes in input.
- Based on the size of the condition number, the function is said to be either ill-conditioned or well-conditioned.
- Using certain planets as reference points can result in ill-conditioned and well-conditioned polynomial systems.
- We will be using Archimedean tropical varieties to estimate condition numbers of polynomial systems.

Condition Number

For a linear system, Ax $=$ b, the condition number (for the change in x under perturbations of b) is $K_A := ||A||||A^{-}1||$ where $||A||$ is the operator norm $supp_{|x|=1}|Ax|$

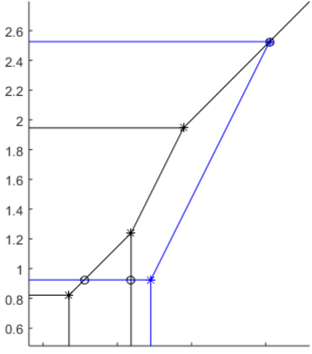


Idea behind Tropical Condition Number

To compute correctly K(F, ζ) exactly requires knowing ζ. Computing $K_{\mathrm{trop}}(F, \zeta)$ exactly requires just a convex hull computation.

Tropical Condition Number

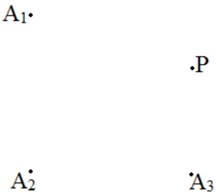
For any 2x2 system of polynomials (f,g), its tropical condition number, at $\zeta \in$ Archtrop(f) $\cap$ Archtrop(g) is just $K_{\mathrm{trop}}(F, \zeta) :=$ K_A where A is the matrix with columns being the generators of the sublattices defined by the lines determined by the 1-cells of archtrop(f) and archtrop(g) containing ζ

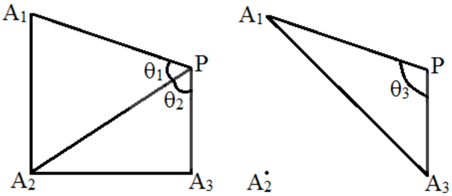


- Goal: Solve polynomial systems of equations for location.
- To do this, we first need to find the polynomials we want to solve.

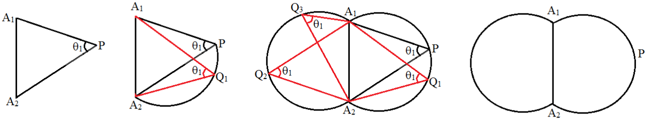
We started in 2D with bivariate quadratic polynomials of the following form:

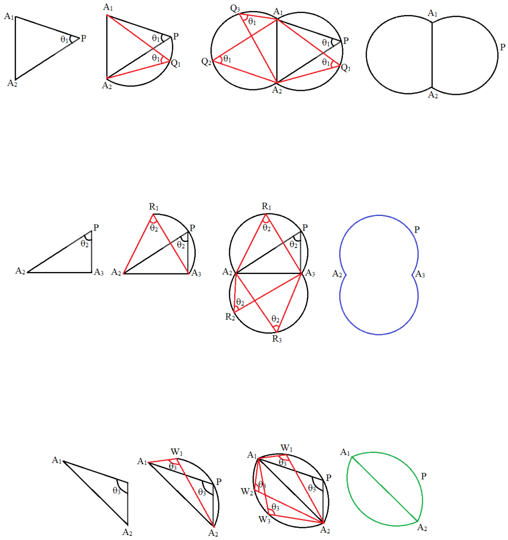
$$f(x, y) = c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2 + c_5 y^2.$$


A1
P
A2
A3

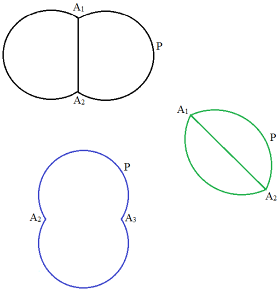

A_1
P
θ_1
θ_2
A_2
A_3
A_1
P
θ_3
A_2
A_3

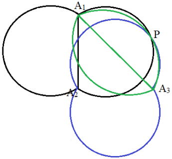
$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$


A1
P
θ1
A2
A1
P
θ1
θ1
Q1
A2
Q3
A1
θ1
P
θ1
Q2
θ1
θ1
Q1
A2
A1
P
A2


A1
A2
P
θ1
Q1
Q2
Q3
R1
R2
R3
A3
θ2
W1
W2
W3
θ3

Goal: Find system of polynomial equations that defines these three semi-algebraic curves!

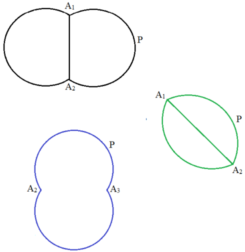


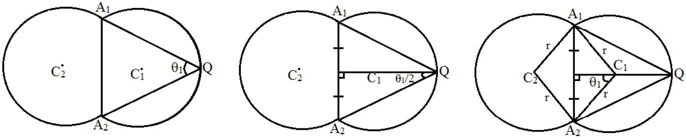


We want to find polynomials of this form:

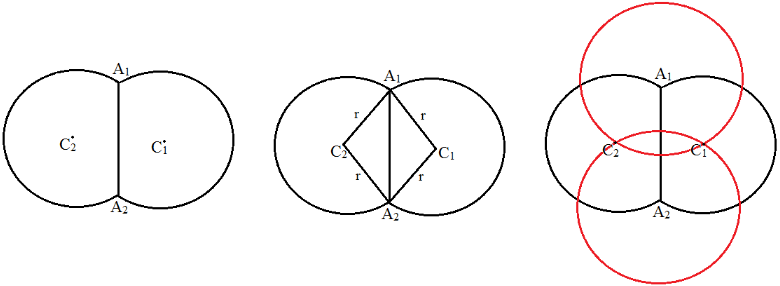
$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

To do this, we will need to find: $C_0 = (x_0, y_0)$ and r.





$$r = (|A_2 - A_1|/2)/sin(\theta_1)$$


A1
A2
C2
C1
A1
A2
C2
C1
r
r
r
r
A1
A2
C2
C1

Example

Set $A_1 = (1,4)$, $A_2 = (1,3)$, and $A_3 = (3,3)$. Set $\theta_1 = 17.1027°$, $\theta_2 = 18.9246°$, and finally $\theta_3 = 36.0274°$.

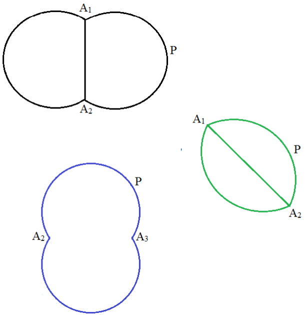
1.0000	3.0000
1.0000	3.0000
3.0000	3.0000

3.0000	3.0000
1.0000	4.0000
1.0000	4.0000

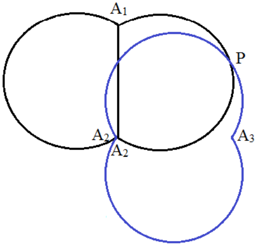
17.1027
18.9246
36.0274

2.0000	6.2500	3.4004
2.0000	-0.2500	3.4004
2.4583	3.5000	1.5417
-0.4583	3.5000	1.5417
2.6875	4.8750	1.9009
1.3125	2.1250	1.9009

$$(x-2.0000)^2+(y-6.2500)^2=3.4004^2$$
$$(x-2.0000)^2+(y+0.2500)^2=3.4004^2$$
$$(x-2.4583)^2+(y-3.5000)^2=1.5417^2$$
$$(x+0.4583)^2+(y-3.5000)^2=1.5417^2$$
$$(x-2.6875)^2+(y-4.8750)^2=1.9009^2$$
$$(x-1.3125)^2+(y-2.1250)^2=1.9009^2$$



$$(x-2.0000)^2+(y-6.2500)^2=3.4004^2$$
$$(x-2.0000)^2+(y+0.2500)^2=3.4004^2$$
$$(x-2.4583)^2+(y-3.5000)^2=1.5417^2$$
$$(x+0.4583)^2+(y-3.5000)^2=1.5417^2$$



$$(x-2.0000)^2+(y-6.2500)^2=3.4004^2$$
$$(x-2.4583)^2+(y-3.5000)^2=1.5417^2$$

$$(x-2.0000)^2+(y-6.2500)^2=3.4004^2$$
$$(x+0.4583)^2+(y-3.5000)^2=1.5417^2$$

$$(x-2.0000)^2+(y+0.2500)^2=3.4004^2$$
$$(x-2.4583)^2+(y-3.5000)^2=1.5417^2$$

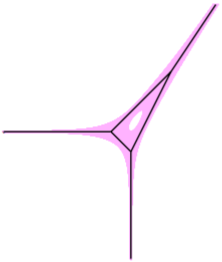
$$(x-2.0000)^2+(y+0.2500)^2=3.4004^2$$
$$(x-2.4583)^2+(y-3.5000)^2=1.5417^2$$

- Goal: Solve the systems of polynomial equations.
- To do this, we need to look at the intersection of ArchTrops of the system.

Why use Archimedean tropical varieties?

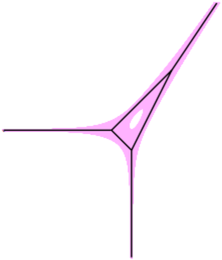
Definition

Let us then define the function $Log|x|$ to be $(log|x_1|, \ldots, log|x_n|)$ and, for any $f \in C[x_1^{\pm 1}, \ldots, x_n^{\pm} 1]$, we define $Amoeba(f)$ to be $\{Log|x| | f(x) = 0, x \in (C^*)^n\}$.



Theorem

For any n-variate t-nomial f and any $\rho \in Amoeba(f)$, there is always a $\sigma \in ArchTrop(f)$ with $|\rho - \sigma| \leq log(t-1)$.



Definition

For a finite set S and a polynomial f written in the form $f(x) = \sum_{i=1}^{t} c_i x^{a_i}$ with $c_i \neq 0$ we define,
$Newt(f) = Conv\{Supp(f)\}$ where $Supp(f) := \{a_i\}_{i \in [t]}$.

Example

$f(x, y) = -4 + 22x + 7y + 12xy - 3x^2 + y^2$
$Newt(f) = Conv\{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (2, 0)\}$

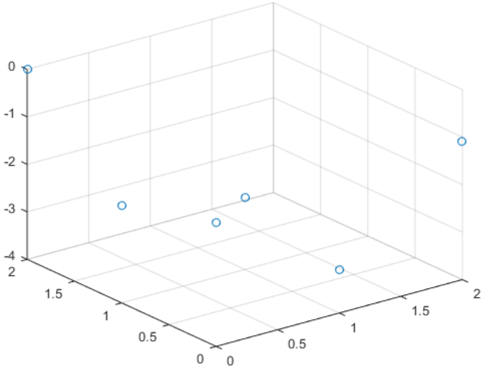
Definition

When the polynomial f is defined as $f(x) = \sum_{i=1}^{t} c_i x^{a_i}$ with $c_i \neq 0$, $ArchNewt(f) := Conv(\{a_i, -log|c_i|\}_{i \in [t]})$.

Example

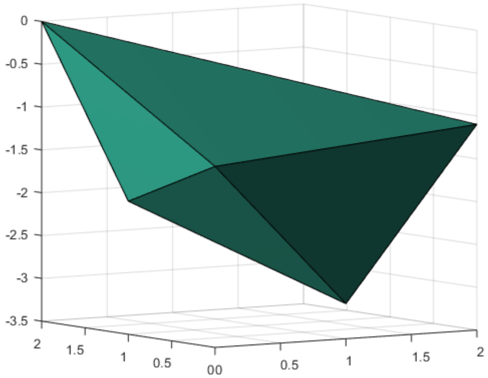
$f(x, y) = -4 + 22x + 7y + 12xy - 3x^2 + y^2$

(0,0,-log$|-4|$)



Definition

When the function f is defined as $f(x) = \sum_{i=1}^{t} c_i x^{a_i}$ with $c_i \neq 0$, $ArchNewt(f) := Conv(\{a_i, -log|c_i|\}_{i\in[t]})$.



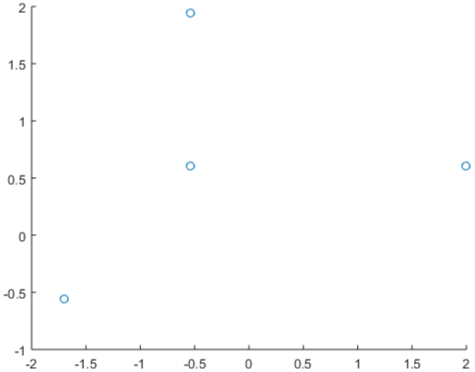
Definition

ArchTrop(f) is defined as the set of all $w \in \mathbb{R}$ with $(w, -1)$ an outer normal of a positive-dimensional face of ArchNewt(f).

Definition

ArchTrop(f) is defined as the set of all $w \in \mathbb{R}$ with $(w, -1)$ an outer normal of a positive-dimensional face of ArchNewt(f).

$$f(x, y) = -4 + 22x + 7y + 12xy - 3x^2 + y^2$$



Definition

ArchTrop(f) is defined as the set of all $w \in \mathbb{R}$ with $(w, -1)$ an outer normal of a positive-dimensional face of ArchNewt(f).

0	-3.6972	0
-1.7047	-0.5596	-1.0000
-0.5754	-2.7726	4.0000
-2.5055	0	0
-0.5390	0.6061	-1.0000
1.9924	0.6061	-1.0000
-0.5390	1.9459	-1.0000
3.8712	3.8712	0

Definition

ArchTrop(f) is defined as the set of all $w \in \mathbb{R}$ with $(w, -1)$ an outer normal of a positive-dimensional face of ArchNewt(f).

-1.7047	-0.5596	-1.0000
-0.5390	0.6061	-1.0000
1.9924	0.6061	-1.0000
-0.5390	1.9459	-1.0000

Definition

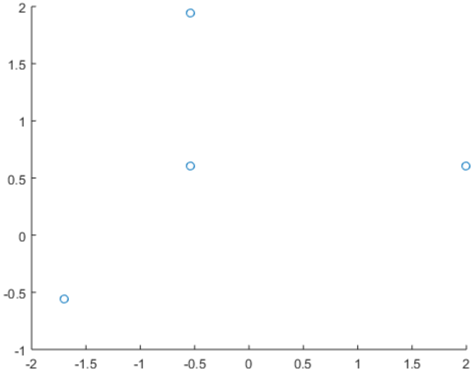
ArchTrop(f) is defined as the set of all $w \in \mathbb{R}$ with $(w, -1)$ an outer normal of a positive-dimensional face of ArchNewt(f).

-1.7047	-0.5596
-0.5390	0.6061
1.9924	0.6061
-0.5390	1.9459

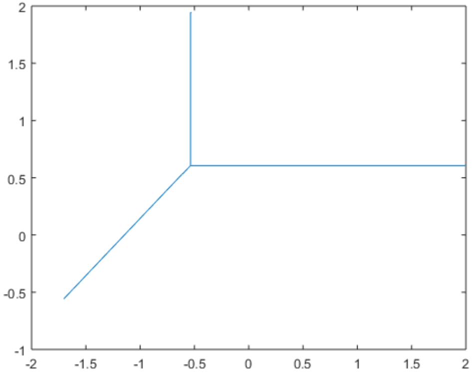
Definition

ArchTrop(f) is defined as the set of all $w \in \mathbb{R}$ with $(w, -1)$ an outer normal of a positive-dimensional face of ArchNewt(f).

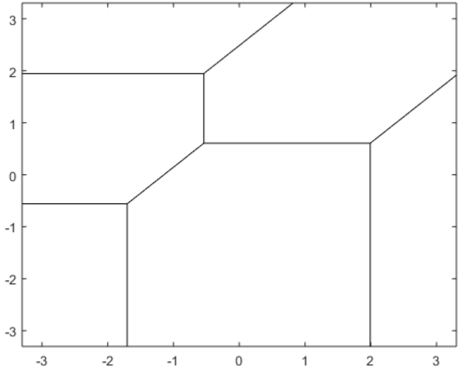
$$f(x, y) = -4 + 22x + 7y + 12xy - 3x^2 + y^2$$



$$f(x, y) = -4 + 22x + 7y + 12xy - 3x^2 + y^2$$



$$f(x,y) = -4 + 22x + 7y + 12xy - 3x^2 + y^2$$



$f(x, y) = -4 + 22x + 7y + 12xy - 3x^2 + y^2$
[-4 22 7 12 -3 1]

$$(x - 2.0000)^2 + (y - 6.2500)^2 = 3.4004^2$$
$$(x - 2.0000)^2 + (y + 0.2500)^2 = 3.4004^2$$
$$(x - 2.4583)^2 + (y - 3.5000)^2 = 1.5417^2$$
$$(x + 0.4583)^2 + (y - 3.5000)^2 = 1.5417^2$$
$$(x - 2.6875)^2 + (y - 4.8750)^2 = 1.9009^2$$
$$(x - 1.3125)^2 + (y - 2.1250)^2 = 1.9009^2$$

31.5000	-4.0000	-12.5000	0	1.0000	1.0000
-7.5000	-4.0000	0.5000	0	1.0000	1.0000
15.9167	-4.9167	-7.0000	0	1.0000	1.0000
10.0833	0.9167	-7.0000	0	1.0000	1.0000
27.3754	-5.3750	-9.7501	0	1.0000	1.0000
2.6246	-2.6250	-4.2499	0	1.0000	1.0000

$$31.5000 - 4.0000x - 12.5000y + 0xy + x^2 + y^2$$
$$-7.5000 - 4.0000x + 0.5000y + 0xy + x^2 + y^2$$
$$15.9167 - 4.9167x - 7.0000y + 0xy + x^2 + y^2$$
$$10.0833 + 0.9167x - 7.0000y + 0xy + x^2 + y^2$$
$$27.3754 - 5.3750x - 9.7501y + 0xy + x^2 + y^2$$
$$2.6246 - 2.6250x - 4.2499y + 0xy + x^2 + y^2$$

Example:

$$f(x,y) = 31.5000 - 4.0000x - 12.5000y + 0xy + x^2 + y^2$$

$$g(x,y) = 15.9167 - 4.9167x - 7.0000y + 0xy + x^2 + y^2$$

Example:

coeffs = [31.5000 -4.0000 -12.5000 0 1.0000 1.0000]

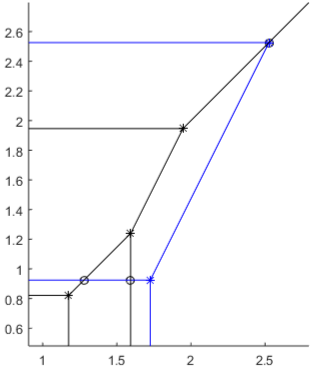
$$f(x, y) = 31.5000 - 4.0000x - 12.5000y + 0xy + x^2 + y^2$$

coeffs = [15.9167 -4.9167 -7.0000 0 1.0000 1.0000]

$$g(x, y) = 15.9167 - 4.9167x - 7.0000y + 0xy + x^2 + y^2$$

$$f(x,y) = 31.5000 - 4.0000x - 12.5000y + 0xy + x^2 + y^2$$

$$g(x,y) = 15.9167 - 4.9167x - 7.0000y + 0xy + x^2 + y^2$$



Intersection points: {(1.2775, 0.9243), (1.5926, 0.9243), (2.5257, While we didn't finish, we've laid the groundwork for solving in 3D.