# ZEROS OF THE MODULAR FORM $\Delta_{k,l} = E_kE_l - E_{k+l}$

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Abstract. We define $\Delta_{k,l}$ to be the modular form $E_kE_l + E_{k+l}$ of weight $k+l$ where $E_k$ is the Eisenstein series of weight $k$ and study the location of zeros of $\Delta_{k,l}$ in $\mathcal{F}$, the standard fundamental domain. We conjecture that all of its zeros are located on the bottom arc of $\mathcal{F}$ and on the lines $x = \pm\frac{1}{2}$.

## 1. Introduction

Rankin and Swinnerton-Dyer proved that all zeros of $E_k$ in the fundamental domain $\mathcal{F}$ lie on the arc $|z| = 1$ [RS]. We study the location of the zeros of the modular form $\Delta_{k,l}$ in $\mathcal{F}$.

**Conjecture 1.1.** *The zeros of $\Delta_{k,l}$ in $\mathcal{F}$ lie on the boundary $\mathcal{B} = \{z = x + iy \in \mathcal{F} | x = \pm\frac{1}{2}$ or $|z| = 1\}$.*

**Conjecture 1.2.** *The modular form $\Delta_{k,l}$ has at least $\lfloor \frac{l}{6} \rfloor - 1$ zeros on the line $x = \frac{1}{2}$.*

**Theorem 1.3.** *The modular form $\Delta_{k,k}$ has at least $\lfloor \frac{k}{6} \rfloor - (1 + n)$ zeros in $\mathcal{F}$ that lie on the line $x = \frac{1}{2}$ where $n$ is the number of zeros of the form $\frac{1}{2} + iy$ for $y > c_0 \frac{\sqrt{k}}{\sqrt{\log k}}$ for $c_0 \leq \frac{1}{\sqrt{8}}$.*

## 2. Background

This material is standard in the theory of modular forms. We use [Z] as reference, while there are many others that would suffice.

The group action of $SL_2(\mathbb{R})$ on $\mathbb{H} = \{z \in \mathbb{C} | \text{Im}(z) > 0\}$ is defined by $z \mapsto \gamma(z)$ where for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$, $\gamma(z) = \frac{az+b}{cz+d}$. We extend this to $\mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$ such that $\gamma(\infty) = \frac{a}{c}$.

A complex-valued function $f$ is a modular form if it is holomorphic for every point $z \in \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$ and satisfies the transformation law $f(\gamma(z)) = f(\frac{az+b}{cz+d}) = (cz + d)^k f(z)$ for all $z \in \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$, all $\gamma \in SL_2(\mathbb{Z})$, and some $k \in \mathbb{Z}$. Typically, $k$ is positive and even since the only modular forms of weight 0 are constant functions, the only modular form of odd weight is the 0-function, and there are no modular forms of negative weight.

Two elements $z_1, z_2 \in \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$ are $SL_2(\mathbb{Z})$-equivalent if there exists some $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma(z_1) = z_2$.

There exist infinitely many $SL_2(\mathbb{Z})$-equivalent regions of $\mathbb{H}$, one being the fundamental domain. This is denoted as $\mathcal{F} = \{z = x + iy \in \mathbb{H} : x \in (-\frac{1}{2}, \frac{1}{2}), |z| \geq 1\}$. If we are concerned with locating the zeros of a modular form, it suffices to locate unique zeros up to $SL_2(\mathbb{Z})$ equivalence. Thus we look for zeros in $\mathcal{F}$. Note that the lines $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ are $SL_2(\mathbb{Z})$-equivalent, as are the two sides of the arc $|z| = 1$, $x \in [-\frac{1}{2}, 0]$ and $|z| = 1$, $x \in [0, -\frac{1}{2}]$ so it suffices to consider only one of each.

The valence formula

$$\frac{1}{2}v_i(f)+\frac{1}{3}v_\rho(f)+\sum_{\substack{z\neq i,\rho\\ z\in\mathbb{H}}}v_z(f)=\frac{k}{12} \tag{2.1}$$

tells us that a modular form $f$ of weight $k$ has precisely $\frac{k}{12}$ zeros.

The Eisenstein series of weight $k$ for $z\in\mathbb{H}\cup\{\infty\}$, $k\geq 4$ is defined by

$$E_k(z)=\frac{1}{2}\sum_{\substack{(c,d)=1\\ c,d\in\mathbb{Z}}}\frac{1}{(cz+d)^k} \tag{2.2}$$

with a corresponding normalized Fourier expansion, $E_k(z)=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi inz}$ where $B_k$ denotes the $k$th Bernoulli number.

## 3. Proof of Theorem 1.3

For $k,l\geq 4$, we focus on the modular form of weight $k+l$, $E_k(z)E_l(z)-E_{k+l}(z)$, and its zeros. Note that this is a cusp form for all $k,l$ and that $E(\frac{1}{2}+iy)\in\mathbb{R}$, so $E_k(\frac{1}{2}+iy)E_l(\frac{1}{2}+iy)-E_{k+l}(\frac{1}{2}+iy)\in\mathbb{R}$ as well. When $k=l=4$, $E_kE_l-E_{k+l}=0$ since $E_4^2=E_8$. Thus for the $k=l$ case, we focus on $k\geq 6$.

We want to approximate $E_k(\frac{1}{2}+iy)^2-E_{2k}(\frac{1}{2}+iy)$ and use the resulting function to exhibit $\lfloor\frac{k}{6}\rfloor$ sign changes, showing that $E_k^2-E_{2k}$ has $\lfloor\frac{k}{6}\rfloor-1$ zeros on the line $x=\frac{1}{2}$. Unfortunately, our method only works up to $y\leq c_0\frac{\sqrt{k}}{\sqrt{\log k}}$ for $c_0\leq\frac{1}{\sqrt{8}}$, so we define $n$ to be the number of zeros of the form $z=\frac{1}{2}+iy$ with $y>c_0\frac{\sqrt{k}}{\sqrt{\log k}}$ of $E_k^2-E_{2k}$. Working with $y$ in our range, we instead prove $\lfloor\frac{k}{6}\rfloor-(1+n)$ zeros.
The points we will use are of the form $z=\frac{1}{2}+iy_m$ where $y_m=\frac{\tan(\theta_m)}{2}$ for $\theta_m=\frac{m\pi}{k}$ where $m\in\mathbb{Z}$ such that $\lceil\frac{k}{3}\rceil\leq m<\frac{k}{2}-n$. If we rewrite $E_k=M_k+R_k$, then $E_k^2-E_{2k}=M_k^2-M_{2k}+2M_kR_k+R_k^2-R_{2k}$. Then we wish to show $|M_k(\frac{1}{2}+iy_m)-M_{2k}(\frac{1}{2}+iy_m)|>|2M_k(\frac{1}{2}+iy_m)R_k(\frac{1}{2}+iy_m)+R_k(\frac{1}{2}+iy_m)^2-R_{2k}(\frac{1}{2}+iy_m)|$. In order to do this, we need to bound $|2M_k(\frac{1}{2}+iy_m)R_k(\frac{1}{2}+iy_m)+R_k(\frac{1}{2}+iy_m)^2-R_{2k}(\frac{1}{2}+iy_m)|$ from above and $|M_k(\frac{1}{2}+iy_m)-M_{2k}(\frac{1}{2}+iy_m)|$ from below, the first of which requires bounding $|R_k|$ on its own.

**Lemma 3.1.** *For all $\frac{\sqrt{3}}{2}\leq y\leq c_0\frac{\sqrt{k}}{\sqrt{\log k}}$ for $c_0\leq\frac{1}{\sqrt{8}}$, the absolute value of the remainder term $|R_k(\frac{1}{2}+iy)|$ is less than $\frac{9+12y}{(\frac{9}{4}+y^2)^{\frac{k}{2}}}$.*

*Proof.* Write $E_k(\frac{1}{2}+iy)=M_k(\frac{1}{2}+iy)+R_k(\frac{1}{2}+iy)$ where

$$M_k(\tfrac{1}{2}+iy)=\underbrace{1+\frac{1}{(\frac{1}{2}+iy)^k}+\frac{1}{(\frac{-1}{2}+iy)^k}}_{c^2+d^2=1,2\text{ except for }(c,d)\text{ where }c=1,d=1\text{ and }c=-1,d=-1} \tag{3.1}$$

and

$$R_k(\tfrac{1}{2}+iy) = \underbrace{\frac{1}{(\frac{3}{2}+iy)^k}}_{c=1,d=1 \text{ and } c=-1,d=-1} + \frac{1}{2}\sum_{\substack{(c,d)=1, c^2+d^2\geq 5\\ c,d\in\mathbb{Z}}} \frac{1}{(c(\frac{1}{2}+iy)+d)^k} \tag{3.2}$$

Then $|R_k(\frac{1}{2}+iy)| = \frac{1}{(\frac{9}{4}+y^2)^{\frac{k}{2}}} + \left|\frac{1}{2}\sum_{\substack{c^2+d^2\geq 5\\ (c,d)=1\\ c,d\in\mathbb{Z}}} \frac{1}{(c(\frac{1}{2}+iy)+d)^k}\right|$.

Rewrite

$$\left|\frac{1}{2}\sum_{\substack{c^2+d^2\geq 5\\ (c,d)=1\\ c,d\in\mathbb{Z}}} \frac{1}{(c(\frac{1}{2}+iy)+d)^k}\right| = \underbrace{\frac{1}{(4+4y^2)^{\frac{k}{2}}} + \frac{1}{(4y^2)^{\frac{k}{2}}} + \frac{1}{(\frac{25}{4}+y^2)^{\frac{k}{2}}}}_{c^2+d^2=5} + \underbrace{\frac{1}{2}\sum_{\substack{c^2+d^2\geq 10\\ (c,d)=1\\ c,d\in\mathbb{Z}}} \frac{1}{((\frac{c}{2}+d)^2+c^2y^2)^{\frac{k}{2}}}}_{T_k(\frac{1}{2}+iy)} \tag{3.3}$$

and observe that $(c,d)$ and $(-c,-d)$ yield identical terms. Then we sum over positive $c$ only, eliminating the coefficient of $\frac{1}{2}$. Similarly, for fixed $c$, the terms for $(c,d)$ and $(c,-(d+c))$ yield idential terms as well. This lets us sum over positive $d$ for each $c$, accounting for the lack of symmetry when $c=1$ and $c=2$. For simplicity, we drop the coprime condition on $c$ and $d$.

Then

$$T_k(\frac{1}{2}+iy) = \sum_{c=1}^{\infty}\sum_{\substack{d\geq 1\\ c^2+d^2\geq 10}}^{\infty} \frac{1}{((\frac{c}{2}+d)^2+c^2y^2)^{\frac{k}{2}}} + \frac{1}{((\frac{c}{2}-d)^2+c^2y^2)^{\frac{k}{2}}} \tag{3.4}$$

and we proceed by finding an upper bound for each fixed $c$. Due to the isolated terms not included in $T_k(\frac{1}{2}+iy)$, $c=1$ and $c=2$ must be bounded separately.

For $c=1$ we have

$$\sum_{\substack{d\geq 1\\ 1+d^2\geq 10}}^{\infty} \frac{1}{((\frac{1}{2}+d)^2+y^2)^{\frac{k}{2}}} + \frac{1}{((\frac{1}{2}-d)^2+y^2)^{\frac{k}{2}}} = \frac{1}{(\frac{5}{2}+y^2)^{\frac{k}{2}}} + 2\sum_{d=3}^{\infty}\frac{1}{((\frac{1}{2}+d)^2+y^2)^{\frac{k}{2}}} \tag{3.5}$$

$$\leq \frac{1}{(\frac{5}{2}+y^2)^{\frac{k}{2}}} + 2\Big(\frac{1}{((\frac{1}{2}+3)^2+y^2)^{\frac{k}{2}}} + \int_3^{\infty}\frac{1}{((\frac{1}{2}+x)^2+y^2)^{\frac{k}{2}}}dx\Big) \tag{3.6}$$

$$\leq \frac{1}{(\frac{5}{2}+y^2)^{\frac{k}{2}}} + 2\Big(\frac{1}{((\frac{1}{2}+3)^2+y^2)^{\frac{k}{2}}} + \int_3^{y+\frac{1}{2}}\frac{1}{((\frac{1}{2}+x)^2+y^2)^{\frac{k}{2}}}dx + \int_{y+\frac{1}{2}}^{\infty}\frac{1}{((\frac{1}{2}+x)^2+y^2)^{\frac{k}{2}}}dx\Big) \tag{3.7}$$

$$< \frac{1}{(\frac{5}{2}+y^2)^{\frac{k}{2}}} + 2\Big(\frac{1}{((\frac{1}{2}+3)^2+y^2)^{\frac{k}{2}}} + \underbrace{\int_3^{y+\frac{1}{2}}\frac{1}{(\frac{1}{2}+3)^2+y^2)^{\frac{k}{2}}}dx}_{\frac{1}{2}+x\leq y} + \underbrace{\int_{y+\frac{1}{2}}^{\infty}\frac{1}{((\frac{1}{2}+x)^2)^{\frac{k}{2}}}dx}_{\frac{1}{2}+x>y}\Big) \tag{3.8}$$

$$< \frac{1}{(\frac{5}{2}+y^2)^{\frac{k}{2}}} + \frac{2+2y}{(\frac{49}{4}+y^2)^{\frac{k}{2}}} + \frac{2}{(k-1)(y+1)^{k-1}} \tag{3.9}$$

Similarly,

$$\sum_{\substack{d\geq 1\\ 4+d^2\geq 10}}^{\infty} \frac{1}{((1+d)^2+4y^2)^{\frac{k}{2}}} + \frac{1}{((1-d)^2+4y^2)^{\frac{k}{2}}} = \underbrace{\frac{1}{(4+4y^2)^{\frac{k}{2}}}}_{d=-3} + \underbrace{\frac{1}{(9+4y^2)^{\frac{k}{2}}}}_{d=-4} + 2\sum_{d=3}^{\infty}\frac{1}{((1+d)^2+4y^2)^{\frac{k}{2}}} \tag{3.10}$$

where

$$2\sum_{d=3}^{\infty}\frac{1}{((1+d)^2+4y^2)^{\frac{k}{2}}} \leq 2\Big(\frac{1}{((1+3)^2+4y^2)^{\frac{k}{2}}}+\int_3^{\infty}\frac{1}{((1+x)^2+4y^2)^{\frac{k}{2}}}dx\Big) \tag{3.11}$$

$$= 2\Big(\frac{1}{(16+4y^2)^{\frac{k}{2}}}+\int_3^{2y-1}\frac{1}{((1+x)^2+4y^2)^{\frac{k}{2}}}dx+\int_{2y-1}^{\infty}\frac{1}{((1+x)^2+4y^2)^{\frac{k}{2}}}dx\Big) \tag{3.12}$$

$$< 2\Big(\frac{1}{(16+4y^2)^{\frac{k}{2}}}+\underbrace{\int_3^{2y-1}\frac{1}{(4y^2)^{\frac{k}{2}}}dx}_{x+1\leq 2y}+\underbrace{\int_{2y-1}^{\infty}\frac{1}{((1+x)^2)^{\frac{k}{2}}}dx}_{x+1>2y}\Big) \tag{3.13}$$

$$< \frac{2}{(16+4y^2)^{\frac{k}{2}}}+\frac{3}{(2y)^{k-1}} \tag{3.14}$$

which totals to $\frac{1}{(4+4y^2)^{\frac{k}{2}}}+\frac{1}{(9+4y^2)^{\frac{k}{2}}}+\frac{2}{(16+4y^2)^{\frac{k}{2}}}+\frac{3}{(2y)^{k-1}}$ for $c=2$.

For general $c \geq 3$,

$$\sum_{\substack{d\geq 1\\ c^2+d^2\geq 10}}^{\infty}\frac{1}{((\frac{c}{2}+d)^2+c^2y^2)^{\frac{k}{2}}}+\frac{1}{((\frac{c}{2}-d)^2+c^2y^2)^{\frac{k}{2}}} = 2\sum_{d=1}^{\infty}\frac{1}{((\frac{c}{2}+d)^2+c^2y^2)^{\frac{k}{2}}} \leq 4\Big(\frac{1}{((\frac{c}{2}+\frac{1-c}{2})^2+c^2y^2)^{\frac{k}{2}}}+\int_{\frac{1-c}{2}}^{\infty}\frac{1}{((\frac{c}{2}+x)^2+c^2y^2)^{\frac{k}{2}}}dx\Big) \tag{3.15}$$

if $c$ is odd, and

$$2\sum_{d=1}^{\infty}\frac{1}{((\frac{c}{2}+d)^2+c^2y^2)^{\frac{k}{2}}} \leq 4\Big(\frac{1}{((\frac{c}{2}+1-\frac{c}{2})^2+c^2y^2)^{\frac{k}{2}}}+\int_{1-\frac{c}{2}}^{\infty}\frac{1}{((\frac{c}{2}+x)^2+c^2y^2)^{\frac{k}{2}}}dx\Big) \tag{3.16}$$

if $c$ is even. We bound odd $c$ by even $c$ to get

$$2\sum_{d=1}^{\infty}\frac{1}{((\frac{c}{2}+d)^2+c^2y^2)^{\frac{k}{2}}} \leq 4\Big(\frac{1}{((\frac{c}{2}+\frac{1-c}{2})^2+c^2y^2)^{\frac{k}{2}}}+\int_{\frac{1-c}{2}}^{\infty}\frac{1}{((\frac{c}{2}+x)^2+c^2y^2)^{\frac{k}{2}}}dx\Big) \tag{3.17}$$

$$< 4\Big(\frac{1}{(\frac{1}{4}+c^2y^2)^{\frac{k}{2}}}+\int_0^{\infty}\frac{1}{((\frac{1}{2}+x)^2+c^2y^2)^{\frac{k}{2}}}dx\Big) \tag{3.18}$$

$$= 4\Big(\frac{1}{(\frac{1}{4}+c^2y^2)^{\frac{k}{2}}}+\Big(\int_0^{cy-\frac{1}{2}}\frac{1}{((\frac{1}{2}+x)^2+c^2y^2)^{\frac{k}{2}}}dx+\int_{cy-\frac{1}{2}}^{\infty}\frac{1}{((\frac{1}{2}+x)^2+c^2y^2)^{\frac{k}{2}}}dx\Big) \tag{3.19}$$

$$< 4\Big(\frac{1}{(\frac{1}{4}+c^2y^2)^{\frac{k}{2}}}+\Big(\underbrace{\int_0^{cy-\frac{1}{2}}\frac{1}{(c^2y^2)^{\frac{k}{2}}}dx}_{x+\frac{1}{2}\leq cy}+\underbrace{\int_{cy-\frac{1}{2}}^{\infty}\frac{1}{((\frac{1}{2}+x)^2)^{\frac{k}{2}}}dx}_{x+\frac{1}{2}>cy}\Big) \tag{3.20}$$

$$< \frac{4}{(\frac{1}{4}+c^2y^2)^{\frac{k}{2}}}+(4+\frac{4}{k-1})\frac{1}{(cy)^{k-1}} \tag{3.21}$$

Summing over all fixed $c \geq 3$ gives us

$$\sum_{c=3}^{\infty}\Big(\frac{4}{(\frac{1}{4}+c^2y^2)^{\frac{k}{2}}}+\frac{4}{(cy)^{k-1}}+\frac{4}{(k-1)(cy)^{k-1}}\Big) < \frac{4}{(\frac{1}{4}+9y^2)^{\frac{k}{2}}}+\frac{4}{(3y)^{k-1}}+\frac{4}{(k-1)(3y)^{k-1}}+\int_3^{\infty}\Big(\frac{1}{(\frac{1}{4}+x^2y^2)^{\frac{k}{2}}}+\frac{1}{(xy)^{k-1}}+\frac{1}{(k-1)(xy)^{k-1}}dx\Big) \tag{3.22}$$

$$< \frac{4}{(\frac{1}{4}+9y^2)^{\frac{k}{2}}}+\frac{4+\frac{4}{k-1}}{(3y)^{k-1}}+\frac{8+\frac{4}{k-1}}{(k-2)3^{k-2}y^{k-1}} < \frac{4}{(\frac{1}{4}+9y^2)^{\frac{k}{2}}}+\frac{11}{(3y)^{k-1}} \tag{3.23}$$

which, combined with our upper bounds for $c=1, c=2$ gives us

$$T_k(\frac{1}{2}+iy) < \frac{1}{(\frac{5}{2}+y^2)^{\frac{k}{2}}}+\frac{2+2y}{(\frac{49}{4}+y^2)^{\frac{k}{2}}}+\frac{2}{(k-1)(y+1)^{k-1}}+\frac{1}{(4+4y^2)^{\frac{k}{2}}}+\frac{1}{(9+4y^2)^{\frac{k}{2}}}+ \frac{2}{(16+4y^2)^{\frac{k}{2}}}+\frac{3}{(2y)^{k-1}}+\frac{4}{(\frac{1}{4}+9y^2)^{\frac{k}{2}}}+\frac{11}{(3y)^{k-1}} \tag{3.24}$$

and

$$|R_k(\tfrac{1}{2}+iy)| < \frac{1}{(\frac{9}{4}+y^2)^{\frac{k}{2}}} + \frac{1}{(4+4y^2)^{\frac{k}{2}}} + \frac{1}{(4y^2)^{\frac{k}{2}}} + \frac{1}{(\frac{25}{4}+y^2)^{\frac{k}{2}}} + \frac{1}{(\frac{5}{2}+y^2)^{\frac{k}{2}}} + \frac{2+2y}{(\frac{49}{4}+y^2)^{\frac{k}{2}}} \tag{3.25}$$

$$+ \frac{2}{(k-1)(y+1)^{k-1}} + \frac{1}{(4+4y^2)^{\frac{k}{2}}} + \frac{1}{(9+4y^2)^{\frac{k}{2}}} + \frac{2}{(16+4y^2)^{\frac{k}{2}}} \tag{3.26}$$

$$+ \frac{3}{(2y)^{k-1}} + \frac{4}{(\frac{1}{4}+9y^2)^{\frac{k}{2}}} + \frac{11}{(3y)^{k-1}} \tag{3.27}$$

$$< \frac{4}{(\frac{9}{4}+y^2)^{\frac{k}{2}}} + \frac{10}{(4y^2)^{\frac{k}{2}}} + \frac{2y}{(\frac{49}{4}+y^2)^{\frac{k}{2}}} + \frac{\frac{2y+2}{k-1}}{(y^2+2y+1)^{\frac{k}{2}}} + \frac{6y}{(4y^2)^{\frac{k}{2}}} + \frac{33y}{(9y^2)^{\frac{k}{2}}} \tag{3.28}$$

$$< \frac{7}{(\frac{9}{4}+y^2)^{\frac{k}{2}}} + \frac{12y+2}{(\frac{49}{4}+y^2)^{\frac{k}{2}}} < \frac{9+12y}{(\frac{9}{4}+y^2)^{\frac{k}{2}}} \tag{3.29}$$

□

Thus for any $z \in \mathbb{H}$ of the form $\frac{1}{2}+iy$, $|R_k(\frac{1}{2}+iy)| < \frac{9+12y}{(\frac{9}{4}+y^2)^{\frac{k}{2}}}$.

**Lemma 3.2.** *For all $\frac{\sqrt{3}}{2} \le y \le c_0\frac{\sqrt{k}}{\sqrt{\log k}}$ for $c_0 \le \frac{1}{\sqrt{8}}$, the absolute value of the main term $|2M_k(\frac{1}{2}+iy)R_k(\frac{1}{2}+iy) + R_k(\frac{1}{2}+iy)^2 - R_{2k}(\frac{1}{2}+iy)|$ is strictly less than $8\left(\frac{9+12y}{(\frac{9}{4}+y^2)^{\frac{k}{2}}}\right)$.*

*Proof.* Recall that $M_k(\frac{1}{2}+iy) = 1 + \frac{1}{(\frac{1}{2}+iy)^k} + \frac{1}{(-\frac{1}{2}+iy)^k}$, so $|M_k(\frac{1}{2}+iy)| \le 1 + |\frac{1}{(\frac{1}{2}+iy)^k}| + |\frac{1}{(-\frac{1}{2}+iy)^k}| \le 3$ and $|R_k(\frac{1}{2}+iy)| < \frac{9+12y}{(\frac{9}{4}+y^2)^{\frac{k}{2}}}$ which is decreasing in $k$. Then

$$|2M_k(\tfrac{1}{2}+iy)R_k(\tfrac{1}{2}+iy) + R_k(\tfrac{1}{2}+iy)^2 - R_{2k}(\tfrac{1}{2}+iy)| \le 2|M_k(\tfrac{1}{2}+iy)||R_k(\tfrac{1}{2}+iy)| + |R_k(\tfrac{1}{2}+iy)|^2 \tag{3.30}$$

$$+ |R_{2k}(\tfrac{1}{2}+iy)| \tag{3.31}$$

$$< 6|R_k(\tfrac{1}{2}+iy)| + 2|R_k(\tfrac{1}{2}+iy)| \tag{3.32}$$

$$= 8|R_k(\tfrac{1}{2}+iy)| \tag{3.33}$$

which implies $2M_k(\frac{1}{2}+iy)R_k(\frac{1}{2}+iy)+R_k(\frac{1}{2}+iy)^2-R_{2k}(\frac{1}{2}+iy) < 8|R_k(\frac{1}{2}+iy)| < 8\left(\frac{9+12y}{(\frac{9}{4}+y^2)^{\frac{k}{2}}}\right)$ by Lemma 3.1. □

**Lemma 3.3.** *For all $\frac{\sqrt{3}}{2} \le y_m = \frac{\tan(\theta_m)}{2} \le c_0\frac{\sqrt{k}}{\sqrt{\log k}}$ for $c_0 \le \frac{1}{\sqrt{8}}$ with $\theta_m = \frac{m\pi}{k}$ where $m \in \mathbb{Z}$ such that $\lceil\frac{k}{3}\rceil < m < \frac{k}{2}-n$, the absolute value of the main term$|M_k(\frac{1}{2}+iy_m)^2 - M_{2k}(\frac{1}{2}+iy_m)^2$ is at least $\frac{4(\frac{1}{4}+y_m^2)^{\frac{k}{2}}-2}{(\frac{1}{4}+y_m^2)^k}$.*

*Proof.* If we rewrite $\frac{1}{2}+iy_m = re^{i\theta_m}$, then

$$M_k(re^{i\theta_m})^2 - M_{2k}(re^{i\theta_m})^2 = (1+\frac{1}{(re^{i\theta_m})^k}+\frac{1}{(re^{i(\pi-\theta_m)})^k})^2 - (1+\frac{1}{(re^{i\theta_m})^{2k}}+\frac{1}{(re^{i(\pi-\theta_m)})^{2k}}) \tag{3.34}$$

$$= \frac{2}{(re^{i\theta_m})^k}+\frac{2}{(re^{i(\pi-\theta_m)})^k}+\frac{2}{(re^{i\theta_m})^k(re^{i(\pi-\theta_m)})^k} \tag{3.35}$$

$$= \frac{2r^k(e^{i(\pi-\theta)k}+e^{i\theta k})+2}{(re^{i\theta k})(r^ke^{i\pi k}e^{-i\theta k})} \tag{3.36}$$

$$= \frac{2r^k(e^{i\pi k}e^{-i\theta k}+e^{i\theta k})+2}{r^{2k}(e^{i\theta k}e^{-i\theta k})} \tag{3.37}$$

$$= \frac{4r^k\cos(\theta k)+2}{r^{2k}} \tag{3.38}$$

and for our points, $\cos(\theta_m k) = \cos(\frac{m\pi}{k}k) = \cos(m\pi) = (-1)^m$ so

$$|M_k(\tfrac{1}{2}+iy_m)^2 - M_{2k}(\tfrac{1}{2}+iy_m)^2| = \left|\frac{4r^k(-1)^m+2}{r^{2k}}\right| \tag{3.39}$$

$$\geq \frac{4r^k-2}{r^{2k}} \tag{3.40}$$

(3.41)

Converting back from polar coordinates gives us $r^k = (\frac{1}{4}+y_m^2)^{\frac{k}{2}}$ so

$$|M_k(\tfrac{1}{2}+iy_m)^2 - M_{2k}(\tfrac{1}{2}+iy_m)^2| \geq \frac{4(\frac{1}{4}+y_m^2)^{\frac{k}{2}}-2}{(\frac{1}{4}+y_m^2)^k} \tag{3.42}$$

□

**Lemma 3.4.** *For all $y_m$ as defined previously, $\frac{4(\frac{1}{4}+y_m^2)^{\frac{k}{2}}-2}{(\frac{1}{4}+y_m^2)^k}$ is strictly greater than $8\left(\frac{9+12y_m}{(\frac{9}{4}+y_m^2)^{\frac{k}{2}}}\right)$.*

*Proof.* We simplify the desired inequality:

$$\frac{4(\frac{1}{4}+y_m^2)^{\frac{k}{2}}-2}{(\frac{1}{4}+y_m^2)^k} > \frac{72+96y_m}{(\frac{9}{4}+y_m^2)^{\frac{k}{2}}} \Rightarrow \frac{1}{(\frac{1}{4}+y_m^2)^{\frac{k}{2}}} - \frac{1}{2(\frac{1}{4}+y_m^2)^{\frac{k}{2}}} > \frac{18+24y_m}{(\frac{9}{4}+y_m^2)^{\frac{k}{2}}} \Rightarrow \left(\frac{\frac{9}{4}+y_m^2}{\frac{1}{4}+y_m^2}\right)^{\frac{k}{2}} > 19+24y_m \tag{3.43}$$

Notice that for all $y_m$ in our range, $(\frac{38}{\sqrt{3}}+24)y_m \geq 19+24y_m$ so we let $c_2 = \frac{38}{\sqrt{3}}+24$ to get

$$\left(\frac{\frac{9}{4}+y_m^2}{\frac{1}{4}+y_m^2}\right)^{\frac{k}{2}} > c_2y_m \tag{3.44}$$

This simplifies further to

$$\frac{k}{2}\log\left(\frac{\frac{9}{4}+y_m^2}{\frac{1}{4}+y_m^2}\right) > \log(c_2y_m) \Rightarrow \frac{k}{2}\log\left(1+\frac{2}{\frac{1}{4}+y_m^2}\right) > \log(c_2y_m) \tag{3.45}$$

and for all $y_m$ in our range, it is the case that $\log(1+\frac{2}{\frac{1}{4}+y_m^2}) \geq \frac{1}{\frac{1}{4}+y_m^2}$.

This gives us

$$k > 2(\tfrac{1}{4}+y_m^2)\log(c_2y_m) \tag{3.46}$$

Since $y_m \leq c_0 \frac{\sqrt{k}}{\sqrt{\log k}}$,

$$2(\tfrac{1}{4} + y_m^2)\log(c_2 y_m) \leq 2(\tfrac{1}{4} + c_0^2 \tfrac{k}{\log k})\log(c_2 c_0 \tfrac{k}{\log k}) \tag{3.47}$$

so we need

$$k > 2(\tfrac{1}{4} + c_0^2 \tfrac{k}{\log k})\log(c_2 c_0 \tfrac{\sqrt{k}}{\sqrt{\log k}}) \tag{3.48}$$

Notice that $c_0 \frac{\sqrt{k}}{\sqrt{\log k}}$ and let $k \geq c_2$. This gives us

$$k > 2(\tfrac{1}{4} + c_0^2 \tfrac{k}{\log k})\log(k^2) = 4(\tfrac{1}{4} + c_0^2 \tfrac{k}{\log k})\log(k) \tag{3.49}$$

which brings us to two cases.
Case 1: If $\frac{1}{4} > \frac{c_0^2 k}{\log k}$, we have $\left(\frac{1}{4} + \frac{c_0^2 k}{\log k}\right) < \frac{1}{2}$ and so

$$k > 4(\tfrac{1}{2})\log(k) \tag{3.50}$$

$$k > 2\log(k) \tag{3.51}$$

which is true for all $k$.
Case 2: If $\frac{1}{4} \leq \frac{c_0^2 k}{\log k}$, then $\left(\frac{1}{4} + \frac{c_0^2 k}{\log k}\right) \leq \frac{2c_0^2 k}{\log k}$ and so

$$k > 4(\tfrac{2c_0^2 k}{\log k})\log(k) = 8c_0^2 k, \tag{3.52}$$

which is true for all $k$ with $c_0 \leq \frac{1}{\sqrt{8}}$.
Thus in both cases, the inequality holds for all $k$, letting us conclude that for all $y_m$ in our range, $\frac{4(\frac{1}{4}+y_m^2)^{\frac{k}{2}}-2}{(\frac{1}{4}+y_m^2)^k}$ is strictly greater than $8\left(\frac{9+12y_m}{(\frac{9}{4}+y_m^2)^{\frac{k}{2}}}\right)$. □

Recall that we set $k \geq c_2$, so the following holds for $k \geq 46 = \lceil c_2 \rceil$. Combining our results from Lemmas 3.2, 3.3, and 3.4, we conclude that $|M_k(\frac{1}{2}+iy_m)^2 - M_{2k}(\frac{1}{2}+iy_m)|$ is strictly greater than $|2M_k(\frac{1}{2}+iy_m)R_k(\frac{1}{2}+iy_m) + R_k(\frac{1}{2}+iy_m)^2 - R_{2k}(\frac{1}{2}+iy_m)|$. This allows us to use $M_k(\frac{1}{2}+iy_m) - M_{2k}(\frac{1}{2}+iy_m)$ as an approximation for $\Delta_{k,k}(\frac{1}{2}+iy_m)$.

From (3.38) we know $M_k(\frac{1}{2}+iy_m)^2 - M_{2k}(\frac{1}{2}+iy_m) = M_k(re^{i\theta_m})^2 - M_{2k}(re^{i\theta_m}) = \frac{4r^k(-1)^m+2}{r^{2k}}$. Since $\lceil \frac{k}{3} \rceil \leq m < \frac{k}{2} - n$ and there are $\frac{k}{6} - n$ integers in $[\lceil \frac{k}{3} \rceil, \frac{k}{2} - n]$, we have shown that $M_k(\frac{1}{2}+iy_m)^2 - M_{2k}(\frac{1}{2}+iy_m)$ exhibits $\frac{k}{6} - n$ sign changes, and thus has $\frac{k}{6} - (1+n)$ zeros. Since this function adequately approximates $\Delta_{k,k}$, it follows that $\Delta_{k,k}$ has $\frac{k}{6} - (1+n)$ zeros on the line $x = \frac{1}{2}$. This concludes our proof.

## 4. Future work: the General Case for $\Delta_{k,l}$

With time, we hope to obtain similar results for the general case of $\Delta_{k,l}$ - when $k \neq l$. Observe that $\Delta_{k,l} = \Delta_{l,k}$ so let us work with $k > l$. If we write $\Delta_{k,l}$ by using our $E_k = M_k + R_k$ substitution, we have $\Delta_{k,l} = M_kM_l + R_kM_l + R_lM_k + R_kR_l - M_{k+l} - R_{k+l}$, with a proposed main term

$$M_k(re^{i\theta})M_l(re^{i\theta}) - M_{k+l}(re^{i\theta}) = \left(\tfrac{r^{2k}+r^k 2\cos(\theta k)}{r^{2k}}\right)\left(\tfrac{r^{2l}+r^l 2\cos(\theta l)}{r^{2l}}\right) - \left(\tfrac{r^{2(k+l)}+r^{(k+l)}2\cos(\theta(k+l))}{r^{2(k+l)}}\right) \tag{4.1}$$

$$= \frac{r^{2l+k}2\cos(\theta k) + r^{2k+l}2\cos(\theta l) + r^{k+l}2\cos(\theta(k-l))}{r^{2(k+l)}} \tag{4.2}$$

Since $k > l$, we suspect that the second term $r^{2k+l}2\cos(\theta l)$ will contribute most to the value of the function. Thus we choose points $re^{i\theta_m}$ for $\theta_m = \frac{m\pi}{l}$ for $\lceil \frac{l}{3} \rceil \leq m < \frac{l}{2}$, analogous to our points in the case of $\Delta_{k,k}$. If we rewrite $k = l + d$, our main term becomes

$$(4.3)\quad \frac{r^{3l+d}2\cos(\theta_m(l+d)) + r^{3l+2d}2\cos(\theta_m l) + r^{2l+d}2\cos(\theta_m d)}{r^{4l+2d)}}$$
$$= \frac{r^{3l+d}2\cos(ml)\cos(\frac{m\pi}{l}d) + r^{3l+2d}2\cos(ml) + r^{2l+d}2\cos(\frac{m\pi}{l}d)}{r^{4l+2d)}}$$
$$= \frac{r^{3l+d}2(-1)^m\cos(\frac{m\pi}{l}d) + r^{3l+2d}2(-1)^m + r^{2l+d}2\cos(\frac{m\pi}{l}d)}{r^{4l+2d}}$$

We would like to show that $|r^{3l+2d}2(-1)^m| > |r^{3l+d}2(-1)^m\cos(\frac{m\pi}{l}d) + r^{2l+d}2\cos(\frac{m\pi}{l}d)|$ by having separate cases for $d \equiv 0, 2, 4 \pmod 6$. This is a result of $\cos(\frac{m\pi}{l}d)$ taking on different values depending on what $d$ is $\pmod 6$. In these three cases, we also want to find a lower bound on $|M_k(\frac{1}{2}+iy_m)M_l(\frac{1}{2}+iy_m) - M_{k+l}(\frac{1}{2}+iy_m)|$.

We believe $|R_kM_l + R_lM_k + R_kR_l - R_{k+l}| < 8|M_l| < 8\left(\frac{9+12y_m}{(\frac{9}{4}+y_m^2)^{\frac{l}{2}}}\right)$. By following the method of proof for $\Delta_{k,k}$, we hope to prove that $\Delta_{k,l}$ has $\lfloor \frac{l}{6} \rfloor - (1+n)$ zeros on the line $x = \frac{1}{2}$, a modified version of Conjecture 1.2. Here, $n$ is the number of zeros of the form $x + iy$ for $y > c_0\frac{\sqrt{l}}{\sqrt{\log l}}$ for $c_0 \leq \frac{1}{\sqrt{8}}$.

This conjecture came from plotting the zeros of $\Delta_{k,l}$ in Mathematica and observing several patterns. Let $B_{k,l}$ be the number of zeros of the form $\frac{1}{2}+iy \in \mathcal{F}$ that $\Delta_{k,l}$. Then we compile a chart of $B_{k,l}$ for $10 \leq k, l \leq 100$, displayed below.

| l\k | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | 50 | 52 | 54 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 |
| 18 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 20 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 22 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 3 | 3 | 2 | 3 |
| 24 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 26 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 28 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 3 | 3 | 4 | 3 | 3 | 4 | 3 | 3 | 4 | 3 | 3 |
| 30 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 32 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 2 | 3 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 34 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 4 | 4 | 5 | 4 | 4 | 5 | 4 | 4 |
| 36 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 4 | 5 | 5 | 4 | 5 | 5 | 5 | 5 | 5 |
| 38 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 3 | 4 | 5 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 40 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 5 | 5 | 6 | 5 | 5 |
| 42 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 6 | 5 | 6 | 6 | 5 | 6 | 6 |
| 44 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 4 | 5 | 6 | 5 | 6 | 6 | 6 | 6 | 6 | 6 |
| 46 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 7 | 6 | 6 |
| 48 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 6 | 7 | 6 | 7 | 7 |
| 50 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 6 | 5 | 6 | 7 | 6 | 7 | 7 | 7 |

| l\k | 56 | 58 | 60 | 62 | 64 | 66 | 68 | 70 | 72 | 74 | 76 | 78 | 80 | 82 | 84 | 86 | 88 | 90 | 92 | 94 | 96 | 98 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 18 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 20 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 22 | 3 | 2 | 3 | 3 | 2 | 3 | 3 | 2 | 3 | 3 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 24 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 26 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 28 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 3 |
| 30 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 32 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 34 | 5 | 4 | 4 | 5 | 4 | 4 | 5 | 4 | 5 | 5 | 4 | 5 | 5 | 4 | 5 | 5 | 4 | 5 | 5 | 4 | 5 | 5 | 4 |
| 36 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 38 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 40 | 6 | 5 | 5 | 6 | 5 | 5 | 6 | 5 | 5 | 6 | 5 | 5 | 6 | 5 | 6 | 6 | 5 | 6 | 6 | 5 | 6 | 6 | 5 |
| 42 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 44 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 46 | 7 | 6 | 6 | 7 | 6 | 6 | 7 | 6 | 6 | 7 | 6 | 6 | 7 | 6 | 6 | 7 | 6 | 6 | 7 | 6 | 7 | 7 | 6 |
| 48 | 6 | 7 | 7 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 50 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |

Each entry corresponds to $B_{k,l}$ for $\Delta_{k,l}$ where the diagonal line connects $B_{k,k}$. We observe that for fixed $l$, $B_{k,l}$ stabilizes to $\lceil \frac{l}{6} \rceil - 1$. The circles correspond to when $B_{k,l}$ stabilizes for $l \equiv 4 \pmod 6$, while the triangles correspond to when $B_{k,l}$ stabilizes for $l \equiv 0 \pmod 6$. This leads us to several patterns that result in conjectures expanding on Conjecture 1.2:

**Conjecture 4.1.** *For fixed* $l \equiv 4 \pmod 6$, $\Delta_{k,l}$ *has* $\lceil \frac{l}{6} \rceil - 1$ *zeros on the line* $x = \frac{1}{2}$ *if* $k \geq k_0$. *Evidence suggests that* $k_0 \leq l + 18(\lfloor \frac{l}{6} \rfloor)$.

**Conjecture 4.2.** *For fixed* $l \equiv 0 \pmod 6$, $\Delta_{k,l}$ *has* $\frac{l}{6} - 1$ *zeros on the line* $x = \frac{1}{2}$ *if* $k \geq l + 4 + 6(\frac{l-1}{6} \pmod 3)$ *or* $k - l \equiv 0, 4 \pmod 6$. *Otherwise,* $\Delta_{k,l}$ *has* $\frac{l}{6} - 2$ *zeros on the line* $x = \frac{1}{2}$.

**Conjecture 4.3.** *For fixed* $l \equiv 2 \pmod 6$*,* $\Delta_{k,l}$ *has* $\lfloor \frac{l}{6} \rfloor - 1$ *zeros on the line* $x = \frac{1}{2}$ *for all* $k \geq l$*.*

We hope to prove a weaker version of Conjecture 1.2, one that is analogous to Theorem 1.3 for general $k, l$:

**Conjecture 4.4.** *The modular form* $\Delta_{k,l}$ *has at least* $\lfloor \frac{l}{6} \rfloor - (1 + n)$ *zeros in* $\mathcal{F}$ *that lie on the line* $x = \frac{1}{2}$ *where* $n$ *is the number of zeros of the form* $\frac{1}{2} + iy$ *with* $y > c_0 \frac{\sqrt{l}}{\sqrt{\log l}}$*.*

Lastly, we would like to find an exact value for $n$. So far, we suspect $n \approx \frac{\sqrt{l}}{6}$. This will give an exact number for how many zeros we can prove the location of, both in the case of $\Delta_{k,k}$ and $\Delta_{k,l}$.

## References


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