Neural Codes: Convexity and Computability

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Motivation

O'Keefe et al. discovered place cells in the 70's (2014 Nobel Prize)

- Place cells encode where an animal is spacially by firing when the animal is in a certain region (approximated by a convex open set).
- More generally, consider a collection $\mathcal{U} = \{U_1, ..., U_n\}$ of open sets in $\mathbb{R}^d$, corresponding to locations where a neuron will fire.
- A neural code describes the sets of neurons that can fire simultaneously, or the intersections of the sets in $\mathcal{U}$.

What is a Neural Code?

Definition

A **neural code** $\mathcal{C}$ is a subset of $2^{[n]}$, and each $\sigma \in \mathcal{C}$ is called a **codeword**. Codewords that are maximal with respect to set inclusion are called **maximal**.

Example

An example of a neural code is $\mathcal{C} = \{\emptyset, \{1,2,3\}, \{1,2\}\{1,4\}\}$.For brevity we write $\mathcal{C} = \{\emptyset, 123, 12, 14\}$. 123 and 14 are the maximal codewords.

We will talk about intersections and size of codewords in the set theoretic sense.

Realization of Neural Codes

Definition

A code $\mathcal{C}$ is **realized** by open sets $U_1, \ldots, U_n$ if

$$\sigma \in \mathcal{C} \iff \bigcap_{i \in \sigma} U_i \setminus \bigcup_{j \notin \sigma} U_j \neq \emptyset$$

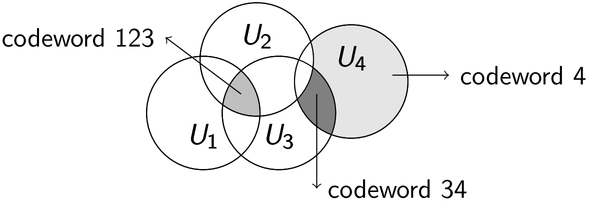


Figure 1: A realization of the code $\mathcal{C} = \{123, 234, 12, 23, 13, 24, 34, 1, 2, 3, 4, \emptyset\}$

Convexity

Definition

A neural code is **convex** *if it can be realized by convex open sets* $U_1, \ldots, U_n$.

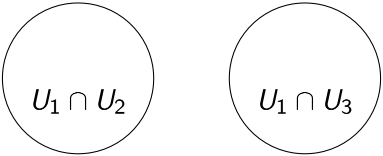


Figure 2: (Nonexample): The code $\mathcal{C} = \{12, 13, \emptyset\}$ is not convex.

The Big Question:

Question

Given a neural code $\mathcal{C}$, is there a way to determine whether $\mathcal{C}$ is convex or not?

- There exist conditions that imply convexity
- There exist conditions that imply non-convexity
- But there are no known necessary **and** sufficient conditions for convexity

Good cover codes

A similar (and strictly weaker) property than convexity is that of being a good cover code.

Definition

A neural code $\mathcal{C}$ is said to be a **good-cover code** *if $\mathcal{C}$ can be realized by contractible open sets $\mathcal{U}$ such that any intersection of sets in $\mathcal{U}$ is also contractible.*

Simplicial Complexes

Definition

An **abstract simplicial complex** *on n vertices is a subset of* $2^{[n]}$ *that is closed under taking subsets. We can topologically realize any simplicial complex (on n vertices) as a subset of the n-simplex in Euclidean space.*

Definition

Given $\mathcal{C}$, *we define* $\Delta(\mathcal{C})$ *to be the smallest simplicial complex containing* $\mathcal{C}$.

Simplicial Complexes

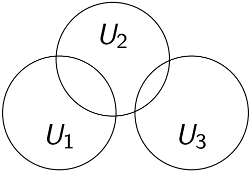
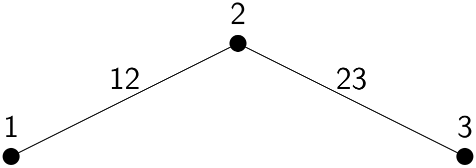


Figure 3: The code $\mathcal{C} = \{12, 23, 1, 2, 3, \emptyset\}$ is convex and a good-cover code. Its simplicial complex, $\Delta(\mathcal{C})$, is realized below.



The link of a simplicial complex

Definition

The **link** *of a face* σ *in a simplicial complex* Δ *is denoted as*

$$Lk_{\sigma}(\Delta) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in \Delta\}.$$

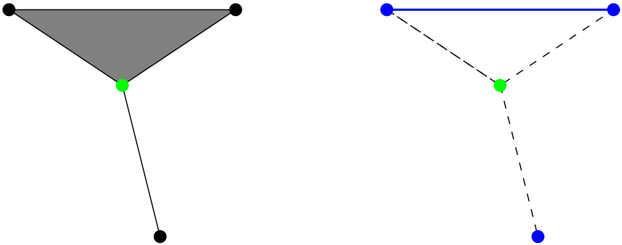


Figure 4: The link of the green vertex in the simplicial complex on the left is shown in blue.

Mandatory Codewords

Definition

Given a simplicial complex Δ, we define $\mathcal{M}(\Delta)$ to be the set of all faces σ that are intersections of maximal faces of Δ and where $Lk_\sigma(\Delta)$ is not contractible. $\mathcal{M}(\Delta)$ is called the set of **mandatory codewords** *for any $\mathcal{C}$ such that $\Delta(\mathcal{C}) = \Delta$.*

The definition is motivated from that fact (not obvious) that any code that does not contain all of its mandatory codewords cannot be convex (or even a good cover code).

Definition

A neural code $\mathcal{C}$ is said to be locally good if it contains all mandatory codewords.

What we already know

Let $\mathcal{C}$ be a neural code. We have the following results:

Theorem (Curto et al.)

If $\mathcal{C}$ is a good cover code, $\mathcal{C}$ is locally good

Note that it follows that if $\mathcal{C}$ is convex, then $\mathcal{C}$ is locally good.

Theorem (Leincamper et al.)

There exists a code that is locally good but not convex.

Locally good = good cover

Theorem (C.)

A code $\mathcal{C}$ is locally good if and only if $\mathcal{C}$ is a good-cover code.

This equates being locally good (which is a strictly local property) with being a good cover code (which means our code has a global “almost convex” realization).

Decision problems for neural codes

One of our main goals to find a characterization for when a neural code is convex. However, is this even possible?

Theorem (C.)

The problem of deciding whether a neural code is locally good is undecidable.

Proof.

- Deciding whether a simplicial complex is contractible is undecidable.
- Outline of reduction: build a neural code based on any simplicial complex so that the code is locally good iff our complex is contractible.

□

A new type of obstruction

Liencamper et al. gave the first counterexample for a neural code that is locally good but not convex. However, the counterexample in question had two alternative ways to resolve the "obstruction."

Theorem (C.)

There exists a locally good nonconvex code $\mathcal{C}$ where $|\Delta(\mathcal{C}) - \mathcal{C}| = 1$.

Because convexity is a monotone property for neural codes with the same simplicial complex, this can be generalized to a new kind of local obstruction.

Future work

The following are several unresolved problems that are closely related to what we have done:

- We say a code is **k-sparse** if for all $\sigma \in \mathcal{C}$, $|\sigma| \leq k$. Are 3-sparse locally good codes convex? (yes for 2, no for 4)
- Is convexity decidable? If so then what is its complexity?
- Finding a necessary and sufficient criterion for convexity.

Acknowledgments

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