A Complexity Bound for the Real Zero Sets of n-variate $(n+4)$-nomials

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Abstract

Given a family of polynomials supported on $\mathcal{A}$, the $\mathcal{A}$-discriminant variety $\nabla_{\mathcal{A}}$ describes the subset of polynomials having a degenerate root. The complement of $Re(\nabla_{\mathcal{A}})$ is a disjoint union of connected components, also referred to as $\mathcal{A}$-discriminant chambers. For a given chamber, the topology of the real zero set of functions with coefficients in that chamber is diffeotopic. Thus, for a given support, an upper bound on the number of discriminant chambers implies an upper bound on the number of diffeotopy types of the real zero set of polynomials with that support. Exploiting homogeneities of the $\mathcal{A}$-discriminant variety and using stratified Morse theory, we prove a $O(n^{24})$ upper bound for the number of discriminant chambers of $\mathcal{A}$ with $\mathcal{A} = n+4$, which in turn implies a bound for the number of diffeotopy types of the real zero set of an n-variate $(n+4)$-nomial.

1 Introduction

While classical algebraic geometry has made immense progress in studying the solution sets of polynomials over algebraically closed fields such as $\mathbb{C}$, the topology of the real zero set of a polynomial may differ dramatically from that of its complex zero set, in particular when its degree far exceeds the number of monomial terms. Characterizing the dependence of the topology of the real zero set of a multivariate polynomial on its monomial structure is an open problem within real algebraic geometry, and we present a polynomial upper bound on the number of diffeotopy types of real zero sets of n-variate $(n+4)$-nomials. In particular, our bounds are independent of the degree of the polynomial.

2 New Topological Upper Bound

The quintessential problem of classifying the topological types of an algebraic set arises in Hilbert's 16th problem, which seeks the determination of all possible nestings of compact components of a real projective plane curve. More generally, one would like to determine the possible diffeotopy types of these curves.

Definition 1. Let $X, Y \subseteq \mathbb{R}^n$. $H : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is an **diffeotopy** if $H(t,\cdot)$ is a diffeomorphism for all $t \in [0,1]$, $H(0,\cdot)|_X = \mathbb{1}_X$, and $H(1,X) = Y$. If such a function exists, we say X and Y are **diffeotopic**.

Observe that diffeotopy is a stronger condition than diffeomorphism, requiring an continuous family of diffeomorphisms that "deform" X to Y.

Definition 2. Given a system of polynomials $F = (f_1, \cdots, f_k)$ in n variables, let $Z^*_{\mathbb{R}}(F)$ (resp. $Z_+(F)$) denote the set of roots of F in $(\mathbb{R}^*)^n$ (resp. $\mathbb{R}^n_+$).

In order to avoid a technicality, we make a genericity assumption on $\mathcal{A}$.

Theorem 1. *Fix $\mathcal{A} \subset \mathbb{Z}^n$ with $\#\mathcal{A} = n+4$, and suppose that the intersection of $\mathcal{A}$ with each facet (i.e. face of dimension $n-1$) of its convex hull consists of exactly n points. Then the number of topological types of the real zero set of f with $Supp(f) = \mathcal{A}$ is $O(n^{24})$.*

For $\#\mathcal{A} = n+1$ (resp. $\#\mathcal{A} = n+2$), it has been shown that there are at most 2 (resp. 3) diffeotopy types of $Z_+(f)$ with $Supp(f) = \mathcal{A}$ [BRS]. That there exist topological bounds – depending only on n and $\#\mathcal{A}$ – is already significant, and was first shown in the 1990s by Lou van den Dries [vdD84]. For the case $\#\mathcal{A} = n+3$, a topological bound which is polynomial in n was found in [DRRS07] and refined in [Rus13]. While upper bounds for the cases $\#\mathcal{A} = n+k$ with $k \geq 4$ may be derived from the results in [GVZ04] or [BV07], the bounds obtained thereof are exponential in n. Thus, our polynomial bound, obtained in a similar fashion to Theorem 1.3 in [DRRS07], is a significant improvement. Our next steps will be to improve our bound further and to attempt to generalize this method to all $5 \leq k \leq n$.

3 Background

3.1 $\mathcal{A}$-discriminants

Definition 3. Given an exponent set $\mathcal{A} = \{a_1, \cdots, a_{n+k}\} \subset \mathbb{Z}^n$, $k \geq 1$, identify with $\mathcal{A}$ the family of n-variate $(n+k)$-nomials

$$\mathscr{F}_{\mathcal{A}} = \left\{ f_c = \sum_{i=1}^{n+k} c_i x^{a_i} \mid c = (c_1, \cdots, c_{n+k}) \in \mathbb{C}P^{n+k-1} \right\}$$

where the notation $x^{a_i} := x_1^{a_{1,i}} \cdots x_n^{a_{n,i}}$ is understood. If the set $\mathcal{A}$ does not lie in some affine $(n-1)$-hyperplane, then f_c is an **honestly** n-variate.

Definition 4. A polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ has a **degenerate root** at ζ if

$$f(\zeta) = \frac{\partial f}{\partial x_1}(\zeta) = \cdots = \frac{\partial f}{\partial x_n}(\zeta) = 0.$$

Definition 5. Given an exponent set $\mathcal{A} = \{a_1, \cdots, a_{n+k}\} \subset \mathbb{Z}^n$, the $\mathcal{A}$**-discriminant variety** is given by

$$\nabla_{\mathcal{A}} = \overline{\{[c_1 : \cdots : c_{n+k}] \in \mathbb{C}P^{n+k-1} \mid f_c \text{ has a degenerate root } \zeta \in (\mathbb{C}^*)^n\}}$$

In the following, we will see that the $\mathcal{A}$-discriminant variety can also be expressed as the zero set of a polynomial in the c_i.

Suppose $f(x), g(x) \in \mathbb{C}[x]$, with $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$. Their Sylvester matrix is given by

$$Syl(f,g) = \begin{bmatrix} a_0 & a_1 & \cdots & a_m & 0 & \\ 0 & \ddots & & & \ddots & 0 \\ & 0 & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & 0 & \\ 0 & \ddots & & & \ddots & 0 \\ & 0 & b_0 & b_1 & \cdots & b_n \end{bmatrix}$$

Let the resultant of f, g be given by $Res(f,g) = \det Syl(f,g)$. Observe that the resultant is a polynomial in the coefficients of f and g.

Theorem 2. *[GKZ94] Suppose $f(x), g(x) \in \mathbb{C}[x]$. Then $Res(f,g) = 0$ if and only if $f(x) = g(x) = 0$ for some $x \in \mathbb{C}$.*

From the previous theorem it follows that a univariate polynomial f has a degenerate root if and only if $Res(f, f') = 0$, and we call the $\mathcal{A}$-discriminant polynomial $\Delta_{\mathcal{A}} = \pm Res(f, f')$. Furthermore, the Cayley trick can be used to generalize the notion of resultants to multivariate polynomials. However, the $\mathcal{A}$-discriminant polynomial is not computationally friendly due to both the number of terms and the size of its coefficients. While $\nabla_{\mathcal{A}}$ is a $n+k-2$-dimensional hypersurface, due to homogeneity we may work with a $k-1$-dimensional slice which we call the reduced $\mathcal{A}$-discriminant variety $\overline{\nabla}_{\mathcal{A}}$. The reduced $\mathcal{A}$-discriminant variety has the advantage that it admits a parametrization of much simpler form than the $\mathcal{A}$-discriminant polynomial known as the Horn-Kapranov Uniformization.

Theorem 3. *When the reduced $\mathcal{A}$-discriminant $\overline{\nabla}_{\mathcal{A}}$ is a hypersurface, it can be parametrized by the* ***Horn-Kapranov Uniformization****:*

$$\overline{\nabla}_{\mathcal{A}} = \overline{\{(B\lambda)^{B^t} \mid \lambda \in \mathbb{C}P^{k-2}\}} \subset (\mathbb{C}^*)^{k-1}$$

where $\hat{\mathcal{A}} = \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+k} \end{bmatrix}$ *and the columns of* $B \in \mathbb{Z}^{(n+k)\times(k-1)}$ *form a basis for the right nullspace of* $\hat{\mathcal{A}}$*.*

By performing column operations on B, we may assume that scalar multiples of the unit vectors $\vec{e}_1, \cdots, \vec{e}_{k-1}$ are among the rows of B. Since $(B\lambda)^{B^t} = 0$ for λ with $\lambda_{k-1} = 0$ and $\{\lambda = [\lambda_1 : \cdots : \lambda_k] \in \mathbb{C}P^{k-1} \mid \lambda_k \neq 0\}$ is homeomorphic to $\mathbb{C}^{k-1}$, the reduced $\mathcal{A}$-discriminant can be expressed as the image of $\{\lambda = (\lambda_1, \cdots : \lambda_{k-2}) \in \mathbb{R}^{k-2} \mid \ell_1(\lambda) \cdots \ell_{n+4}(\lambda) \neq 0\}$ under the mapping:

$$\Psi(\lambda) = \left(\prod_{i=1}^{n+k} \ell_i^{b_{m,i}}(\lambda) \right)_{m=1}^{k-1}$$

where $b_1, \cdots, b_{k-1}$ are the columns of the matrix B and $\ell_i(\lambda) = \sum_{j=1}^{k-2} b_{j,i}\lambda_j + b_{k-1,i}$.

Definition 6. $Re(\overline{\nabla}_{\mathcal{A}})^c$ consists of a finite number of connected components called reduced **discriminant chambers**.

Theorem 4. *[DRRS07] If f_1 and f_2 represent polynomials in the same reduced $\mathcal{A}$-discriminant chamber, then $Z^*_{\mathbb{R}}(f_1)$ and $Z^*_{\mathbb{R}}(f_2)$ are diffeotopic.*

This is significant because a bound on the number of discriminant chambers implies a bound on the number of topological types of the real zero set of polynomials in the family $\mathscr{F}_{\mathcal{A}}$.

3.2 Critical Points Method

Definition 7. Suppose X is a Whitney stratified space. Let the **critical point** of a smooth function $f : X \to \mathbb{R}$ be a critical point of the restriction of f to a stratum of X. f is a **Morse function** on X if its critical values are distinct and each critical point of f is nondegenerate, i.e. the Hessian of f at each critical point is nonsingular.

Theorem 5. *Suppose X is a Whitney stratified space and f is a Morse function with adjacent critical values $c_0 < c_1$. As c varies within (c_0, c_1), the topological type of $f^{-1}(c)$ is constant. [Hir97]*

Critical Points Method. Suppose $X \subset \mathbb{R}^n$ admits a Whitney stratification such that each stratum of X has codimension ≥ 1, and suppose that the coordinate projection π_n when restricted to X has finitely many critical values. Let M_n be the number of critical points of the restriction of π_n to X, and inductively define

$$\begin{aligned} M_k &= \max_{\substack{c_j \neq c_i^j \\ j=k+1,\cdots,n}} \left(\#\text{ critical values of } \pi_k|_{X\cap\{x_k=c, x_{k+1}=c_{k+1},\cdots,x_n=c_n\}}\right) \\ c_1^k, \cdots, c_{M_k}^k &= \text{ critical values of } \pi_k \end{aligned}$$

for $k = n-1, \cdots, 1$. Assume that $\pi_k|_{X\cap\{x_k=c, x_{k+1}=c_{k+1},\cdots,x_n=c_n\}}$ has finitely many critical values for each k, so M_k is finite. Then the number of connected components of $\mathbb{R}^n \setminus X$ is bounded above by

$$\prod_{i=1}^{n} (M_i + 1).$$

Proof sketch To count the number of connected components of $(\mathbb{R})^n \setminus X$, let us introduce hyperplanes $H_1, \cdots, H_{M_n}$ at the x_n coordinates corresponding to the critical values of $\pi_n|_X$. Since each connected component of $T = \mathbb{R}^n \setminus (X \cup H_1 \cup \cdots \cup H_{M_n})$ is contained in a connected component of $\mathbb{R}^n \setminus X$, it is sufficient to count the connected components of T. Since the hyperplanes $H_1, \cdots, H_{M_n}$ divide $\mathbb{R}^n$ into $1 + M_n$ 'slabs' (of the form $(a,b) \times \mathbb{R}^{n-1}$), it is sufficient to bound the number of components of $(slab) \setminus X$. By Theorem 5, this is equivalent to bounding the number of connected components of $\{x \in \mathbb{R}^n \mid x_n = c \neq c_1^n, \cdots, c_{M_n}^n\} \cap X$. The result follows from induction on dimension.

3.3 Sheared Binomial Systems

Definition 8. Let $l_1, \cdots, l_j \in \mathbb{R}[\lambda_1, \cdots, \lambda_k]$ be polynomials of degree ≤ 1. For all $(b_{1,1}, \cdots, b_{1,j}), \cdots, (b_{k,1}, \cdots, b_{k,j}) \in \mathbb{R}^j$ linearly independent ($j \geq k$), call the equations given by

$$S = \begin{pmatrix} \prod_{i=1}^{j} l_i^{b_{1,i}}(\lambda) \\ \vdots \\ \prod_{i=1}^{j} l_i^{b_{k,i}}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

a $k \times k$ **sheared binomial system with** j **factors** (referred to as a **Gale Dual System** in [BS07]). Also call each l_i a **factor** of the system.

Definition 9. With the notation from above, suppose that $a_1, \cdots, a_k \in \mathbb{Z}^k$ are vectors such that $\mathcal{B} = \{(a_1, b_1), \cdots, (a_k, b_k)\} \subset \mathbb{Z}^k \oplus \mathbb{Z}^j$ is a linearly independent collection, and let $h_1, \cdots, h_j \in \mathbb{R}[\lambda_1, \cdots, \lambda_k]$ be a collection of polynomials of degree $\leq d$. Call the equations given by

$$\lambda^{a_i} h(\lambda)^{b_i} = 1 \text{ for } i = 1, \cdots, k$$

a (d, k)**-dense Gale system** and call each h_i a **factor** of the system.

Theorem 6. *1. The number of non-degenerate roots $\lambda \in \mathbb{R}^k$ of any $k \times k$ sheared binomial system with $n + k$ factors is bounded above by:*

(a) $n + 1$ for $k = 1$

(b) $\frac{e^4+3}{4} 2^{\binom{k}{2}} n^k$ for $k > 1$ [BS07].

2. [RSS11] The number of non-degenerate positive roots of a (d, ℓ)-dense Gale system with n factors is bounded above by

$$\frac{e^2 + 3}{4} 2^{\binom{\ell}{2}} n^\ell d^\ell$$

4 Proof of Theorem 1

In the following, the critical points method and a bound on the number of real roots of polynomials with a certain structure are used to bound number of chambers in complement of the reduced $\mathcal{A}$-discriminant of an n-variate $(n + 4)$-nomial. Because components of $\overline{\nabla}_{\mathcal{A}}$ not contained in the image of the Horn-Kapranov parametrization have codimension > 1, it is sufficient to consider the image of the map:

$$\Psi(\lambda) = \left(\prod_{i=1}^{n+4} l_i^{b_{1,i}}(\lambda), \prod_{i=1}^{n+4} l_i^{b_{2,i}}(\lambda), \prod_{i=1}^{n+4} l_i^{b_{3,i}}(\lambda) \right)$$

where b_1, b_2, b_3 are the columns of the matrix B, $l_i(\lambda) = b_{1,i}\lambda_1 + b_{2,i}\lambda_2 + b_{3,i}$, and Ψ is defined on $\{\lambda \in \mathbb{R}^2 \mid l_1(\lambda) \cdots l_{n+4}(\lambda) \neq 0\}$. Let Ω denote the closure of the image of Ψ.

Sign combinations: Since each ℓ_i has constant sign on one 'side' of its zero set (which is a line in the plane), counting the number of possible sign combinations of the vector $(\ell_1(\lambda), \cdots, \ell_{n+4}(\lambda))$ is equivalent to counting the number of components of $\mathbb{R}^2$ in the complement of $n + 4$ lines, which is given by [Sch01]

$$1 + (n + 4) + \binom{n+4}{2} = \frac{1}{2}(n^2 + 9n + 22)$$

Notation: Denote $\frac{\partial \psi_i}{\partial \lambda_j}$ by ψ_{ij} and let π_i be the projection of $\mathbb{R}^k$ onto the ith coordinate ($i \leq k$). Let $\mathcal{M}_3(\mathcal{A})$ be the number of critical points of π_3 restricted to Ω.

- Critical points occuring in one 'sheet' of Ω occur when

$$[0] = D(\pi_3 \circ \Psi)(\lambda) = D\pi_3(\Psi(\lambda))D\Psi(\lambda) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} D\Psi(\lambda) \implies \psi_{31} = \psi_{32} = 0$$

By the product rule and after dividing out by $\psi_3(\lambda)$, we have $\sum_{i=1}^{n+4} b_{3,i}b_{1,i}(l_i(\lambda))^{-1} = \sum_{i=1}^{n+4} b_{3,i}b_{2,i}(l_i(\lambda))^{-1} = 0$. Multiplying through by $l_1(\lambda)\cdots l_{n+4}(\lambda)$, we obtain 2 equations in two variables of degree $\leq n+3$ which, by Bézout's theorem, implies there are at most $(n+3)^2$ solutions. Observe that this count also includes cusp-like singularities, which occur when the tangent space at λ is a degenerate plane.

- Critical points arising from the intersection of two 'sheets' of $\overline{\nabla}_{\mathcal{A}}$: These are pairs $\lambda, \lambda' \in \mathbb{R}^2$ satisfying

$$\Psi(\lambda) = (\psi_1(\lambda), \psi_2(\lambda), \psi_3(\lambda)) = (\psi_1(\lambda'), \psi_2(\lambda'), \psi_3(\lambda')) = \Psi(\lambda') \qquad \text{and}$$
$$\pi_3\left((\Psi_1(\lambda) \times \Psi_2(\lambda)) \times (\Psi_1(\lambda') \times \Psi_2(\lambda'))\right) = 0$$

which is a $(4(n+3), 2)$-dense system with $2(n+4)+1$ factors. Counting sign combinations, there are at most

$$\frac{e^2+3}{4} 2^{\binom{2}{2}} (2n+9)^2 (4n+12)^2 \frac{1}{4}(n^2+9n+11)^2$$

isolated solutions.

- Critical points arising from the intersection of 3 'sheets' of $\overline{\nabla}_{\mathcal{A}}$: These are triples $\lambda, \lambda', \lambda'' \in \mathbb{R}^2$ such that $\Psi(\lambda) = \Psi(\lambda') = \Psi(\lambda'')$, which is a 6×6 sheared system with $3(n+4) = (3n+6)+6$ factors. Since we may permute $\{\lambda, \lambda', \lambda''\}$, there are at most

$$\frac{1}{3!} \cdot \frac{e^2+3}{4} 2^{\binom{6}{2}} (3n+6)^6 \frac{1}{8}(n^2+9n+22)^3 = (e^2+3)2^9 3^5 (n+2)^6 (n^2+9n+22)^3$$

critical points arising from triple intersections.

- x- and y-plane Intersections: Consider when two sheets of Ω intersect the plane $\{x=0\}$. This occurs for $\lambda, \lambda' \in \mathbb{R}^2$ such that $\Psi(\lambda) = \Psi(\lambda')$ and $\psi_1(\lambda) = \psi_1(\lambda') = 0$, which occurs when $\ell_i(\lambda) = 0$ for i in a set of cardinality of at most $n+3$ (since $b_{3,i} > 0$ for at most $n+3$ many i). This is a $(1,4)$-dense system with $n+5$ factors having at most $\frac{e^2+3}{4} 2^{\binom{4}{2}} (n+5)^4$ solutions. Doing the same for the set $\{y=0\}$ and accounting for sign combinations and permutations of λ, λ', there are at most $\frac{e^2+3}{4} 2^{\binom{4}{2}} (n+5)^4 (n+3) \frac{1}{4}(n^2+9n+22)^2$ solutions. Other intersections may arise for $\lambda \in \mathbb{R}^2$ with $\psi_2(\lambda) = \psi_3(\lambda) = 0$, which has at most $\binom{n+4}{2}$ solutions. Finally, nodes in the plane $\{x=0\}$ occur for $\lambda \in \mathbb{R}^2$ with $\psi_1(\lambda) = 0$ and $(\Psi_1(\lambda) \times \Psi_2(\lambda)) \times \vec{e}_1 = 0$, which is equivalent to solving one of at most $n+3$ many systems of polynomials of degree 1 and $2(n+3)$. Since we may do the same for the plane $\{y=0\}$, there are at most $4(n+3)^2$ solutions.

Therefore, we have

$$\mathcal{M}_3(\mathcal{A}) \leq (n+3)^2 + (e^2+3)(2n+9)^2(n+3)^2(n^2+9n+11)^2 + (e^2+3)2^9 3^5 (n+2)^6 (n^2+9n+22)^3$$
$$+ (e^2+3)2^2(n+5)^4(n+3)(n^2+9n+22)^2 + \frac{(n+4)(n+3)}{2} + 4(n+2)^2$$

Next, fix c not equal to one of the critical values identified above, and consider $\Omega \cap \{(x,y,z) \in \mathbb{R}^3 \mid z = c\}$. Let $\mathcal{M}_2(\mathcal{A})$ be the number of critical points of π_2 restricted to this set.

- Critical points: This occurs for $\lambda \in \mathbb{R}^2$ such that $\psi_3(\lambda) = c$ and the intersection of the tangent space at λ and the hyperplane defined by $z=0$ has y-component $=0$:

$$0 = \pi_2((\Psi_1(\lambda) \times \Psi_2(\lambda)) \times \vec{e}_3)$$

Observe that this also includes cusps, which occur intersection of the tangent space at λ with the hyperplane defined by $z=0$ is degenerate. This is a $(2(n+3), 2)$-dense system with $n+5$ factors and therefore at most $\frac{e^2+3}{4} 2^{\binom{2}{2}} (n+5)^2 (2(n+3))^2 \frac{1}{2}(n^2+9n+22)$ solutions.

- Intersections: When two $\lambda, \lambda' \in \mathbb{R}^2$ such that $\Psi(\lambda) = \Psi(\lambda')$ and $\psi_3(\lambda) = \psi_3(\lambda') = c$? This is a 4×4 sheared system, so accounting for permutations of λ, λ', we have at most

$$\frac{1}{2} \cdot \frac{e^2+3}{4} 2^{\binom{4}{2}} (2n+4)^4 \frac{1}{4}(n^2+9n+22)^2 = (e^2+4)2^3(n+2)^4(n^2+9n+22)^2$$

solutions.

- x-axis Intersections: This occurs for λ such that $\psi_1(\lambda) = 0$ and $\psi_1(\lambda) = c$. The former occurs when $\ell_i(\lambda) = 0$ for some i (out of $n+3$ possible). This is a $(1,2)$-dense system with $n+5$ factors. Accounting for sign combinations, there are at most $\frac{e^2+3}{4}(n+5)^2(n+3)\frac{1}{2}(n^2+9n+22)$ solutions.

Thus, we have

$$\mathcal{M}_2(\mathcal{A}) \leq \frac{e^2+3}{4}\left(2(n+5)^2(n+3)^2(n^2+9n+22) + 2(n+2)^4(n^2+9n+22)^2 + (n+5)^2(n+3)\frac{1}{2}(n^2+9n+22)\right)$$

Next, fix c' not equal to one of the critical values identified above. Leting $\mathcal{M}_1(\mathcal{A})$ be the number of critical points of π_1 restricted to $\Omega \cap \{(x,y,z) \in \mathbb{R}^3 \mid y = c', z = c\}$, we see that this is equal to the number of intersections of the line defined by $y = c', z = c$ and Ω. This is equivalent to counting the number of $\lambda \in \mathbb{R}^2$ such that $\psi_2(\lambda) = c'$ and $\psi_3(\lambda) = c$, which is a 2×2 sheared system with $n+4$ factors. Accounting for sign combinations of $(\ell_1(\lambda), \cdots, \ell_{n+4}(\lambda))$ and at most one intersection with the plane $\{x = 0\}$, we have

$$\mathcal{M}_1(\mathcal{A}) \leq \frac{e^2+3}{4}2^{\binom{2}{2}}(n+2)^2\frac{1}{2}(n^2+9n+22)+1$$

solutions.

Taken together, the number of discriminant chambers is bounded above by

$$(\mathcal{M}_1(\mathcal{A})+1)(\mathcal{M}_2(\mathcal{A})+1)(\mathcal{M}_3(\mathcal{A})+1) = O(n^{24})$$

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