NONVANISHING OF HECKE L–SERIES AND ℓ-TORSION IN CLASS GROUPS

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1. Introduction and statement of results

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \mod 4$. Let $\mathcal{O}_K$ be the ring of integers, $\mathrm{Cl}(K)$ be the ideal class group, $h(-D)$ be the class number, and $\varepsilon(n) = (-D/n) = (n/D)$ be the Kronecker symbol associated to K. We view ε as a quadratic character of $(\mathcal{O}_K/\sqrt{-D}\mathcal{O}_K)^\times$ via the isomorphism

$$\mathbb{Z}/D\mathbb{Z} \cong \mathcal{O}_K/\sqrt{-D}\mathcal{O}_K.$$

Let ψ_k be a Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_K$ satisfying

$$\psi_k(\alpha\mathcal{O}_K) = \varepsilon(\alpha)\alpha^{2k-1} \quad \text{for} \quad (\alpha\mathcal{O}_K, \sqrt{-D}\mathcal{O}_D) = 1, \quad k \in \mathbb{Z}^+. \tag{1.1}$$

One can use (1.1) to show that ψ_k satisfies (see [R4])

$$\overline{\psi_k(\mathfrak{a})} = \psi_k(\overline{\mathfrak{a}}) \quad \text{for ideals } \mathfrak{a} \text{ prime to } \sqrt{-D}\mathcal{O}_K. \tag{1.2}$$

Next, let $d \equiv 1 \mod 4$ be a squarefree integer relatively prime to D. Then $(d/N(\cdot))$ is a primitive Hecke character of K of conductor $d\mathcal{O}_K$, and

$$\psi_{d,k} := (d/N(\cdot))\psi_k$$

is the Hecke character of K of conductor $d\sqrt{-D}\mathcal{O}_K$ given by the quadratic twist of ψ_k by $(d/N(\cdot))$. Clearly, $\psi_{d,k}$ also satisfies (1.2). To ease notation, we will sometimes write $\psi = \psi_{d,k}$.

Let $\Psi_{d,k}(D)$ be the set of all such Hecke characters ψ. Then $\#\Psi_{d,k}(D) = h(-D)$, and if ψ_0 is any such character then

$$\Psi_{d,k}(D) = \{\psi_0\xi : \ \xi \in \widehat{\mathrm{Cl}(K)}\}.$$

The L–series of ψ is defined by

$$L(\psi, s) := \sum_{\mathfrak{a}} \psi(\mathfrak{a})N(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) > k + \frac{1}{2}$$

where the sum is over nonzero integral ideals $\mathfrak{a}$ of K. The L–series $L(\psi, s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 2k - s$ with central value $L(\psi, k)$ and root number

$$W(\psi) = (-1)^{k-1}\mathrm{sign}(d)(-1)^{\frac{D+1}{4}}. \tag{1.3}$$

The Hecke characters ψ are examples of "canonical" Hecke characters in the sense of Rohrlich [R2]. These characters of great arithmetic interest. For example, the canonical Hecke characters were first studied by Gross [G], who constructed a "canonical" elliptic $\mathbb{Q}$-curve $A(D)$ associated to $\psi \in \Psi_{1,1}(D)$. In particular, he showed that the extended Hecke character $\chi_H := \psi \circ N_{H/K}$ of the Hilbert class field H of K corresponds to a unique (up to H-isogeny) $\mathbb{Q}$-curve $A(D)/H$ whose L–series factorizes as

$$L(A(D)/H, s) = L(\chi_H, s)L(\overline{\chi_H}, s) = \prod_{\psi \in \Psi_{1,1}(D)} L(\psi, s)L(\overline{\psi}, s).$$

Gross conjectured that

$$\mathrm{rank}(A(D)(H)) = \begin{cases} 0, & D \equiv 7 \mod 8 \\ 2h(-D), & D \equiv 3 \mod 8. \end{cases}$$

Because the conjecture predicts an *exact* formula for the rank, the curves $A(D)/H$ form an important test case for the Birch and Swinnerton-Dyer conjecture. Gross's conjecture is known due to the works [G, R1, R2, MR, MY]. More generally, a canonical Hecke character $\psi \in \Psi_{d,k}(D)$ corresponds to a p-adic Galois representation A_ψ, and one can study the order of the associated Bloch-Kato p-Selmer group $\mathrm{Sel}_p(A_\psi/K)$.

We denote the number of nonvanishing central values in the family $\Psi_{d,k}(D)$ by

$$NV_{d,k}(D) := \#\{\psi \in \Psi_{d,k}(D) : \ L(\psi, k) \neq 0\}.$$

Moreover, note that the Galois group $G_k := \mathrm{Gal}(\overline{\mathbb{Q}}/K(\zeta_{2k-1}))$ acts on $\Psi_{d,k}(D)$ by

$$\psi \mapsto \psi^\sigma, \quad \sigma \in G_k.$$

The nonvanishing of the central values $L(\psi, k)$ was studied in [R1, R2, MR, Y, RVY, MY, LX] under the assumption that G_k acts transitively on $\Psi_{d,k}(D)$. In particular, by work of Shimura [Shi], this implies that if $L(\psi, k) \neq 0$ for some $\psi \in \Psi_{d,k}(D)$, then $NV_{d,k}(D) = h(-D)$. On the other hand, if G_k does *not* act transitively, then the existence of one nonvanishing central value no longer implies that all of the central values are nonvanishing. It is therefore of interest to understand how $NV_{d,k}(D)$ grows as $D \to \infty$.

Let $K/\mathbb{Q}$ be a number field of discriminant D_K and degree n, and let $\mathrm{Cl}_\ell(K)$ be the ℓ-torsion subgroup of the ideal class group $\mathrm{Cl}(K)$. Assuming the Generalized Riemann Hypothesis (GRH), Ellenberg and Venkatesh [EV] proved the non-trivial bound

$$\#\mathrm{Cl}_\ell(K) \ll_{n,\epsilon} |D_K|^{\frac{1}{2} - \frac{1}{2\ell(n-1)} + \epsilon}. \tag{1.4}$$

The second author [M] used this bound to prove that

$$NV_{d,k}(D) \gg_\epsilon D^{\frac{1}{2(2k-1)} - \epsilon}. \tag{1.5}$$

Very recently, Ellenberg, Pierce, and Wood [EPW] combined results in [EV] with a new sieve method (which they call the "Chebyshev" sieve) to prove that (1.4) holds unconditionally for $n \leq 5$, up to an exceptional set of discriminants with natural density zero. In this paper, we will combine the works [M, EPW] to prove an asymptotic formula with a power-saving error term for the number of discriminants D for which (1.5) holds unconditionally. In particular, will prove that (1.5) holds unconditionally for 100% of imaginary quadratic fields within certain families.

In order to state our main results, we fix the following assumptions and notation.

Fix a pair (d, k) such that $\mathrm{sign}(d) = (-1)^{k-1}$. Let $\mathcal{S}_{d,k}$ be the set of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ such that $D \equiv 7 \mod 8$, all prime divisors of d split in K, and D is either prime or coprime to $2k - 1$. For $X > 0$ define the following subsets of $\mathcal{S}_{d,k}$:

$$\mathcal{S}_{d,k}(X) := \{K \in \mathcal{S}_{d,k} : D \leq X\}$$

and

$$\mathcal{S}_{d,k}^{\mathrm{NV}}(X) := \{K \in \mathcal{S}_{d,k}(X) : \ NV_{d,k}(D) \gg_\epsilon D^{\frac{1}{2(2k-1)} - \epsilon}\}.$$

Remark 1.1. The conditions on D in the definition of $\mathcal{S}_{d,k}$ are technical conditions needed for the proofs. For example, the congruence $D \equiv 7 \mod 8$ ensures that the root number $W(\psi) = 1$ for all $\psi \in \Psi_{d,k}(D)$, and the splitting condition ensures that Heegner points of discriminant $-D$ exist on the modular curve $X_0(4d^2)$.

Our main result is the following asymptotic formula with a power-saving error term.

Theorem 1.2. *Given the prime factorizations $d = \prod_{i=1}^{m} p_i$ and $2k-1 = \prod_{i=1}^{n} q_i^{a_i}$, we have*

$$\#\mathcal{S}_{d,k}^{\mathrm{NV}}(X) = 2^{-m}\left(1 - \prod_{i=1}^{n}\left(\frac{1}{1+q_i}\right)\right)\prod_{i=1}^{m}\left(\frac{1}{1+p_i^{-1}}\right)\frac{1}{6\zeta(2)}X + O_{d,k}(X^{1-\frac{1}{2(2k-1)}}) \tag{1.6}$$

as $X \to \infty$.

Corollary 1.3. *We have*

$$\frac{\#\mathcal{S}_{d,k}^{\mathrm{NV}}(X)}{\#\mathcal{S}_{d,k}(X)} = 1 + O_{d,k}(X^{-\frac{1}{2(2k-1)}}) \tag{1.7}$$

as $X \to \infty$. In particular, the bound (1.5) holds for 100% of imaginary quadratic fields $K \in \mathcal{S}_{d,k}$.

An important component of the proof of Theorem 1.2 is an effective way of producing at least one nonvanishing central value *without* assuming that G_k acts transitively on $\Psi_{d,k}(D)$. We will prove the following effective nonvanishing theorem.

Theorem 1.4. *Fix a pair (d,k) such that $d \equiv 1 \mod 4$ is squarefree and sign$(d) = (-1)^{k-1}$. Let $D \equiv 7 \mod 8$ be such that all prime divisors of d split in $K = \mathbb{Q}(\sqrt{-D})$. Then if $D > 64d^4(k+1)^4$, there exists at least one $\psi \in \Psi_{d,k}(D)$ such that $L(\psi,k) \neq 0$.*

To prove Theorem 1.4, we will use a variation on the geometric approach in [BD] which is based on the position of Heegner points in the cusp at infinity of a modular curve. This notion of "quantification in the cusp" using Heegner points to prove nonvanishing theorems originated in [MV], and has since been employed in many other instances.

2. Nonvanishing of half-integral weight theta series

Fix a pair (d,ℓ) where $d \equiv 1 \mod 4$ is a squarefree integer and $\ell \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer such that $\mathrm{sign}(d) = (-1)^{\ell}$. Define the theta series

$$\theta_{d,\ell}(z) := (2y)^{-\ell/2}\sum_{(n,d)=1}\left(\frac{d}{n}\right)H_\ell(n\sqrt{2y})e(n^2z), \quad z = x+iy \in \mathbb{H}, \quad e(z) := e^{2\pi i z}$$

where $H_\ell(x)$ is the degree ℓ Hermite polynomial

$$H_\ell(x) := \frac{1}{(\sqrt{8\pi})^\ell}\sum_{j=0}^{\lfloor \ell/2\rfloor}\frac{\ell!}{j!(\ell-2j)!}(-1)^j(\sqrt{8\pi}x)^{\ell-2j}.$$

The theta series $\theta_{d,\ell}(z)$ is a weight $\ell+\frac{1}{2}$ modular form for $\Gamma_0(4d^2)$ (see []).

To prove Theorem 1.4, we will need the following effective zero-free region for $\theta_{d,\ell}(z)$ which is of independent interest.

Proposition 2.1. *If $y = \mathrm{Im}(z) > (\ell+2)^2$, then $\theta_{d,\ell}(z) \neq 0$.*

The following inequalities will be used in the proof of Proposition 2.1.

Lemma 2.2. *For $x > \ell$ we have*

$$\left(\frac{8\pi-2}{8\pi-1}\right)x^\ell \leq H_\ell(x) \leq x^\ell.$$

Proof. First write

$$H_\ell(x) = \sum_{j=0}^{\lfloor \ell/2\rfloor}\frac{\ell!}{j!(\ell-2j)!}(-1)^j\frac{x^{\ell-2j}}{(8\pi)^j} = x^\ell - \frac{\ell!}{(\ell-2)!8\pi}x^{\ell-2} + x^\ell\sum_{j=2}^{\lfloor \ell/2\rfloor}c_{\ell,j},$$

where

$$c_{\ell,j} := \frac{\ell!}{j!(\ell-2j)!}\frac{(-1)^j}{(8\pi)^j x^{2j}}.$$

Now, for $x \geq \ell$ we have the bound

$$\left|\frac{c_{\ell,j+1}}{c_{\ell,j}}\right| = \frac{(\ell-2j)(\ell-2j-1)}{(j+1)8\pi x^2} \leq \frac{\ell^2}{8\pi x^2} \leq \frac{1}{8\pi}.$$

Then it follows that

$$\begin{aligned}
H_\ell(x) &\leq x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} + \sum_{j=2}^{\infty} |c_{\ell,j}|\right] \\
&\leq x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} + \frac{\ell!}{(\ell-2)!8\pi x^2}\sum_{j=1}^{\infty}\left(\frac{1}{8\pi}\right)^j\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} + \frac{\ell!}{(\ell-2)!8\pi x^2}\left(\frac{1}{8\pi-1}\right)\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2}\left(\frac{8\pi-2}{8\pi-1}\right)\right] \\
&\leq x^\ell.
\end{aligned}$$

On the other hand, arguing similarly with the reverse triangle inequality, for $x \geq \ell$ we have

$$\begin{aligned}
H_\ell(x) &\geq x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} - \sum_{j=2}^{\infty} |c_{\ell,j}|\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} - \frac{\ell!}{(\ell-2)!8\pi x^2}\sum_{j=1}^{\infty}\left(\frac{1}{8\pi}\right)^j\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2}\sum_{j=0}^{\infty}\left(\frac{1}{8\pi}\right)^j\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2}\left(\frac{8\pi}{8\pi-1}\right)\right] \\
&= x^\ell \left[1 - \frac{\ell(\ell-1)}{(8\pi-1)x^2}\right] \\
&\geq x^\ell \left[1 - \frac{\ell^2}{(8\pi-1)x^2}\right] \\
&\geq x^\ell \left[1 - \frac{1}{(8\pi-1)}\right] \\
&= \left(\frac{8\pi-2}{8\pi-1}\right) x^\ell.
\end{aligned}$$

□

Lemma 2.3. *If* $t > (\ell+2)^2$ *then*

$$t - \frac{\ell}{4\pi}\log(\pi t) > \frac{\ell+1}{\pi}\log(2) \tag{2.1}$$

and

$$t - \frac{\ell}{12\pi}\log(t) > \frac{3\ell+2}{12\pi}\log(2). \tag{2.2}$$

Proof. We first consider the inequality (2.1). Clearly, we see that

$$t - \frac{\ell}{4\pi}\log(\pi t) > \frac{\ell+1}{\pi}\log(2) \quad \Longleftrightarrow \quad 4\pi t - \ell\log(16\pi t) > 4\log 2.$$

Moreover, the function $g_\ell(t) := 4\pi t - \ell\log(16\pi t)$ is strictly increasing for $t > \frac{\ell}{4\pi}$. Hence, if we assume that $t > (\ell+2)^2 > \frac{\ell}{4\pi}$, then we have

$$\begin{aligned} g_\ell(t) > g_\ell((\ell+2)^2) &= 4\pi(\ell+2)^2 - \ell\log(16\pi(\ell+2)^2) \\ &= 16\pi + \ell(16\pi - \log 16\pi) + \ell(4\pi\ell - 2\log(\ell+2)) \\ &> 4\log 2. \end{aligned}$$

On the other hand, the inequality (2.2) follows from (2.1) since

$$t - \frac{\ell}{12\pi}\log(t) > t - \frac{\ell}{4\pi}\log(\pi t) > \frac{\ell+1}{\pi}\log(2) > \frac{3\ell+2}{12\pi}\log(2).$$

□

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Using the definition of $H_\ell(x)$ and the Kronecker symbol $\left(\frac{d}{n}\right)$, along with the condition $\mathrm{sign}(d) = (-1)^\ell$, for $n \neq 0$ we have

$$\left(\frac{d}{-n}\right) H_\ell(-n\sqrt{2y}) = \mathrm{sign}(d)\left(\frac{d}{n}\right)(-1)^\ell H_\ell(n\sqrt{2y}) = (-1)^{2\ell}\left(\frac{d}{n}\right) H_\ell(n\sqrt{2y}) = \left(\frac{d}{n}\right) H_\ell(n\sqrt{2y}).$$

Then the theta series can be written as

$$\theta_{d,\ell}(z) = (2y)^{-\ell/2}\left[\left(\frac{d}{0}\right) H_\ell(0) + 2\sum_{n=1}^{\infty}\left(\frac{d}{n}\right) H_\ell(n\sqrt{2y})e(n^2 z)\right]. \tag{2.3}$$

From here forward we assume that $y > (\ell+2)^2$. We will consider the cases $d = 1$ and $d \neq 1$ separately.

Case 1 ($d = 1$): If $d = 1$, then

$$\left|\left(\frac{1}{0}\right) H_\ell(0)\right| = \frac{\ell!}{(8\pi)^{\ell/2}\,(\ell/2)!}.$$

Therefore, by (2.3) if

$$\sum_{n=1}^{\infty} |H_\ell(n\sqrt{2y})e(n^2 z)| < \frac{\ell!}{2(8\pi)^{\ell/2}\,(\ell/2)!},$$

then the reverse triangle inequality implies that $\theta_{1,\ell}(z) \neq 0$.

Consider the function

$$f_\ell(t) := \frac{\log(2^{t-1}t^\ell)}{2\pi(t^2-1)}.$$

Since $f_\ell(t)$ is strictly decreasing for $t > 1$, we have

$$y > (\ell+2)^2 > f_\ell(2) \geq f_\ell(n), \quad n \geq 2.$$

The inequality $y > f_\ell(n)$ is equivalent to

$$\frac{n^\ell}{e^{2\pi(n^2-1)y}} < 2^{1-n}, \quad n \geq 2. \tag{2.4}$$

We can now estimate the series as

$$\begin{aligned}\sum_{n=1}^{\infty}|H_\ell(n\sqrt{2y})e(n^2z)| &\le (2y)^{\ell/2}\sum_{n=1}^{\infty}\frac{n^\ell}{e^{2\pi n^2 y}}\\ &= (2y)^{\ell/2}\frac{1}{e^{2\pi y}}\sum_{n=1}^{\infty}\frac{n^\ell}{e^{2\pi(n^2-1)y}}\\ &\le (2y)^{\ell/2}\frac{1}{e^{2\pi y}}\sum_{n=1}^{\infty}2^{1-n}\\ &= (2y)^{\ell/2}\frac{2}{e^{2\pi y}}\\ &< \frac{1}{2(8\pi)^{\ell/2}}\\ &\le \frac{\ell!}{2(8\pi)^{\ell/2}\,(\ell/2)!},\end{aligned}$$

where the first inequality follows from the upper bound in Lemma 2.2 (since $y > (\ell+2)^2$ we have $n\sqrt{2y} \ge \ell$ for all $n \ge 1$), the second inequality follows from (2.4), and a short calculation shows that the third inequality is equivalent to Lemma 2.3, inequality (2.1). This proves Case 1.

Case 2 ($d \ne 1$): Since $d \ne 1$, we have $\left(\frac{d}{0}\right) = 0$, and (2.3) can be written as

$$\theta_{d,\ell}(z) = 2(2y)^{-\ell/2}\left[H_\ell(\sqrt{2y})e(z) + \sum_{n=2}^{\infty}\left(\frac{d}{n}\right)H_\ell(n\sqrt{2y})e(n^2z)\right].$$

Therefore, if

$$\sum_{n=2}^{\infty}|H_\ell(n\sqrt{2y})e(n^2z)| < |H_\ell(\sqrt{2y})e(z)|,$$

then the reverse triangle inequality implies that $\theta_{d,\ell}(z) \ne 0$.

We have $H_0(\sqrt{2y}) = 1$ and $H_1(\sqrt{2y}) = \sqrt{2y} > 1$. Moreover, if $\ell \ge 2$ then by the lower bound in Lemma 2.2 we have

$$H_\ell(\sqrt{2y}) \ge \frac{8\pi-2}{8\pi-1}(2y)^{\ell/2} > 1.$$

Hence it suffices to show that

$$\sum_{n=2}^{\infty}|H_\ell(n\sqrt{2y})e(n^2z)| < |e(z)| = \frac{1}{e^{2\pi y}}.$$

A modification of the argument in Case 1 shows that

$$\sum_{n=2}^{\infty}|H_\ell(n\sqrt{2y})e(n^2z)| \le (2y)^{\ell/2}\frac{2^{\ell+1}}{e^{8\pi y}} < \frac{1}{e^{2\pi y}},$$

where a short calculation shows that the second inequality in equivalent to Lemma 2.3, inequality (2.2). This proves Case 2. □

3. Proof of Theorem 1.4

Fix a pair (d,k) where $d \equiv 1 \mod 4$ is a squarefree integer and $k \in \mathbb{Z}^+$ is a positive integer such that $\mathrm{sign}(d) = (-1)^{k-1}$. Consider the C^∞ function $F_{d,k} : \mathbb{H} \to \mathbb{R}_{\ge 0}$ defined by

$$F_{d,k}(z) := \mathrm{Im}(z)^{k-\frac{1}{2}}|\theta_{d,k-1}(z)|^2.$$

Since $\theta_{d,k-1}$ is a weight $k-\frac{1}{2}$ modular form for $\Gamma_0(4d^2)$, the function $F_{d,k}$ is $\Gamma_0(4d^2)$-invariant.

Let $D \equiv 7 \mod 8$ be a positive integer such that all prime divisors of d split in $K = \mathbb{Q}(\sqrt{-D})$. Then Heegner points of discriminant $-D$ exist on the modular curve $X_0(4d^2) := \Gamma_0(4d^2)\backslash\mathbb{H}$. In particular, we can fix a square root $r \mod 8d^2$ of $-D \mod 16d^2$, and for any primitive integral ideal $\mathfrak{a} \subset \mathcal{O}_K$ we can write

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}\left(\frac{-b+\sqrt{-D}}{2}\right), \quad a = N_{K/\mathbb{Q}}(\mathfrak{a}), \quad b \in \mathbb{Z},$$

where $b \equiv r \mod 8d^2$ and $b^2 \equiv -D \mod 16ad^2$. Then

$$\tau_{[\mathfrak{a}]}^{(r)} = \frac{-b+\sqrt{-D}}{8ad^2} \in \mathbb{H}$$

defines a Heegner point on $X_0(4d^2)$ which depends only on the ideal class $[\mathfrak{a}]$ and on $r \mod 8d^2$.

Define the $\mathrm{Cl}(K)$-orbit of Heegner points

$$\mathcal{O}_{D,4d^2,r} := \{\tau_{[\mathfrak{a}]}^{(r)} : [\mathfrak{a}] \in \mathrm{Cl}(K)\}.$$

Then by [KMY, Theorem 3.5], we have the following exact formula for the average of the central values

$$\frac{1}{h(-D)} \sum_{\psi\in\Psi_{d,k}(D)} L(\psi,k) = c(k)\frac{\pi}{\sqrt{D}} \sum_{[\mathfrak{a}]\in\mathrm{Cl}(K)} F_{d,k}(\tau_{[\mathfrak{a}]}^{(r)}), \tag{3.1}$$

where $c(k) := 2(8\pi)^{k-1}/(k-1)!$. This formula is independent of the choice of r.

Now, let $\mathfrak{a} = \mathcal{O}_K$ (so that $N_{K/\mathbb{Q}}(\mathfrak{a}) = 1$) and write

$$\tau := \tau_{[\mathcal{O}_K]}^{(r)}\frac{-b+\sqrt{-D}}{8d^2}.$$

Since $F_{d,k}$ is nonnegative, we have

$$\sum_{\psi\in\Psi_{d,k}(D)} L(\psi,k) \geq \pi c(k)\frac{h(-D)}{\sqrt{D}}F_{d,k}(\tau).$$

By Proposition 2.1, if $\mathrm{Im}(\tau) > (k+1)^2$ then $F_{d,k}(\tau) > 0$. But

$$\mathrm{Im}(\tau) = \frac{\sqrt{D}}{8d^2} > (k+1)^2 \quad \Longleftrightarrow \quad D > 64d^4(k+1)^4.$$

In particular, we have shown that if $D > 64d^4(k+1)^4$, then

$$\sum_{\psi\in\Psi_{d,k}(D)} L(\psi,k) > 0,$$

which implies that there exists at least one $\psi \in \Psi_{d,k}(D)$ such that $L(\psi,k) \neq 0$. This completes the proof of Theorem 1.4. $\square$

4. Proofs of Theorem 1.2 and Corollary 1.3

In this section we prove Theorem 1.2 and Corollary 1.3.

For convenience, we recall the setup from the introduction. Fix a pair (d,k) such that $\mathrm{sign}(d) = (-1)^{k-1}$. Let $\mathcal{S}_{d,k}$ be the set of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ such that $D \equiv 7 \mod 8$, all prime divisors of d split in K, and D is either prime or coprime to $2k-1$. For $X > 0$ define the following subsets of $\mathcal{S}_{d,k}$:

$$\mathcal{S}_{d,k}(X) := \{K \in \mathcal{S}_{d,k} : D \leq X\}$$

and

$$\mathcal{S}_{d,k}^{\mathrm{NV}}(X) := \{K \in \mathcal{S}_{d,k}(X) : \ NV_{d,k}(D) \gg_\epsilon D^{\frac{1}{2(2k-1)}-\epsilon}\}.$$

In addition, we will need the subset

$$\mathcal{S}_{d,k}^{\mathrm{Tor}}(X) := \{K \in \mathcal{S}_{d,k}(X) : \text{ the bound (1.4) holds with } n = 2 \text{ and } \ell = 2k-1\}.$$

We begin by giving asymptotic formulae with power-saving error terms for $\#\mathcal{S}_{d,k}(X)$ and $\#\mathcal{S}_{d,k}^{\mathrm{Tor}}(X)$.

Proposition 4.1. *Given the prime factorizations $d = \prod_{i=1}^m p_i$ and $2k-1 = \prod_{i=1}^n q_i^{a_i}$, we have*

$$\#\mathcal{S}_{d,k}(X) = 2^{-m}\left(1 - \prod_{i=1}^n \left(\frac{1}{1+q_i}\right)\right)\prod_{i=1}^m \left(\frac{1}{1+p_i^{-1}}\right)\frac{X}{6\zeta(2)} + O_k\left(2d\left(\prod_{i=1}^n q_i\right)X^{\frac{1}{2}}\right)$$

and

$$\#\mathcal{S}_{d,k}^{\mathrm{Tor}}(X) = 2^{-m}\left(1 - \prod_{i=1}^n \left(\frac{1}{1+q_i}\right)\right)\prod_{i=1}^m \left(\frac{1}{1+p_i^{-1}}\right)\frac{X}{6\zeta(2)} + O_k(X^{1-\frac{1}{2(2k-1)}})$$

as $X \to \infty$.

Proof. First we decompose the set $\mathcal{S}_{d,k}(X)$ into the disjoint union

$$\mathcal{S}_{d,k}(X) = \{K \in \mathcal{S}_{d,k}(X) : (D, 2k-1) = 1\} \sqcup \{K \in \mathcal{S}_{d,k}(X) : D \text{ is prime and } (D, 2k-1) \neq 1\}.$$

The set on the right hand side of this decomposition consists of the prime divisors $p \leq X$ of $2k-1$. Therefore, if we let $\mathcal{S}_{d,k}^1(X)$ denote the set on the left hand side of this decomposition, then

$$\#\mathcal{S}_{d,k}(X) = \#\mathcal{S}_{d,k}^1(X) + t(2k-1; X),$$

where $t(2k-1; X)$ denotes the number of prime divisors $p \leq X$ of $2k-1$.

We will need the following result of Ellenberg, Pierce and Wood [EPW, Proposition 8.1] which counts quadratic number fields of bounded discriminant with prescribed local conditions.

Proposition 4.2. *Let P be a finite set of primes of K. For each $p \in P$ we choose a splitting type at p and assign a corresponding density as follows:*

$$\begin{aligned} \delta_p &:= \frac{1}{2}(1+p^{-1})^{-1} && \textit{if } p \textit{ splits} \\ \delta_p &:= \frac{1}{2}(1+p^{-1})^{-1} && \textit{if } p \textit{ is inert} \\ \delta_p &:= (1+p)^{-1} && \textit{if } p \textit{ is ramified} \end{aligned}$$

Let $e = \prod_{p\in P} p$ and $\delta_e = \prod_{p \in P}\delta_p$. Let $N_2^{\pm}(X; P)$ be the number of real (respectively imaginary) quadratic extensions of $\mathbb{Q}$ with fundamental discriminant $|D_K| \leq X$ such that for each $p \in P$ with prescribed splitting type in K as above, then we have

$$N_2^{\pm}(X; P) = \frac{\delta_e}{2\zeta(2)}X + O(eX^{\frac{1}{2}}).$$

In order to use Proposition 4.2 to count $\mathcal{S}_{d,k}^1(X)$, we must further decompose this set into subsets satisfying appropriate local conditions.

Note that the condition $D \equiv 7 \mod 8$ is equivalent to having the prime 2 split in K, and the condition $(D, 2k-1) = 1$ is equivalent to having all prime divisors of $2k-1$ unramified in K.

Now, consider the prime factorizations $d = \prod_{i=1}^m p_i$ (recall that d is squarefree) and $2k-1 = \prod_{i=1}^n q_i^{a_i}$. Let $\mathcal{S}_{d,k}^2(X)$ be the subset of all $K \in \mathcal{S}_{d,k}^1(X)$ such that the primes in the set

$$P_d := \{2, p_1, \ldots, p_m\}$$

split in K. Similarly, let $\mathcal{S}^3_{d,k}(X)$ be the subset of all $K \in \mathcal{S}^1_{d,k}(X)$ such that the primes in the set P_d split in K and the primes in the set

$$Q_k := \{q_1, \dots, q_n\}$$

ramify in K. Then by the preceding observations, we have

$$\mathcal{S}^1_{d,k}(X) = \mathcal{S}^2_{d,k}(X) \setminus \mathcal{S}^3_{d,k}(X),$$

so that

$$\#\mathcal{S}^1_{d,k}(X) = \#\mathcal{S}^2_{d,k}(X) - \#\mathcal{S}^3_{d,k}(X).$$

Define the set of primes $R_{d,k} := P_d \cup Q_k$. Then by Proposition 4.2, we have

$$\#\mathcal{S}^2_{d,k}(X) = N^-(X; P_d) = 2^{-m} \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}} \right) \frac{X}{6\zeta(2)} + O(2dX^{\frac{1}{2}})$$

and

$$\#\mathcal{S}^3_{d,k}(X) = N^-(X; R_{d,k}) = 2^{-m} \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}} \right) \prod_{i=1}^{n} \left(\frac{1}{1+q_i} \right) \frac{X}{6\zeta(2)} + O\left(2d \left(\prod_{i=1}^{n} q_i \right) X^{\frac{1}{2}} \right).$$

It follows that

$$\#\mathcal{S}^1_{d,k}(X) = 2^{-m} \left(1 - \prod_{i=1}^{n} \left(\frac{1}{1+q_i} \right) \right) \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}} \right) \frac{X}{6\zeta(2)} + O\left(2d \left(\prod_{i=1}^{n} q_i \right) X^{\frac{1}{2}} \right).$$

Then using that

$$t(2k-1; X) \ll_k 1,$$

we get the asymptotic formula

$$\#\mathcal{S}_{d,k}(X) = 2^{-m} \left(1 - \prod_{i=1}^{n} \left(\frac{1}{1+q_i} \right) \right) \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}} \right) \frac{X}{6\zeta(2)} + O_k\left(2d \left(\prod_{i=1}^{n} q_i \right) X^{\frac{1}{2}} \right).$$

Next, let $\mathcal{S}^{\neg\mathrm{Tor}}_{d,k}(X)$ denote the subset of all $K \in \mathcal{S}_{d,k}(X)$ which *fail* to satisfy the bound (1.4) with $\ell = 2k-1$, and write

$$\#\mathcal{S}^{\mathrm{tor}}_{d,k}(X) = \#\mathcal{S}_{d,k}(X) - \#\mathcal{S}^{\neg\mathrm{Tor}}_{d,k}(X).$$

As a consequence of [EPW, Theorem 1], we have

$$\begin{aligned}\#\mathcal{S}^{\neg\mathrm{Tor}}_{d,k}(X) &\ll \#\{\text{quadratic fields } K/\mathbb{Q} \text{ with } |D_K| \le X \text{ which fail to satisfy (1.4) with } \ell = 2k-1\} \\ &\ll X^{1-\frac{1}{2(2k-1)}}. \end{aligned} \tag{4.1}$$

This gives the asymptotic formula

$$\#\mathcal{S}^{\mathrm{Tor}}_{d,k}(X) = 2^{-m} \left(1 - \prod_{i=1}^{n} \left(\frac{1}{1+q_i} \right) \right) \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}} \right) \frac{X}{6\zeta(2)} + O_k(X^{1-\frac{1}{2(2k-1)}}).$$

□

Proof of Theorem 1.2. Let

$$c_{d,k} := 64d^4(k+1)^4$$

be the constant appearing in Theorem 1.4. Then for $X \gg c_{d,k}$, we decompose the set $\mathcal{S}^{\mathrm{Tor}}_{d,k}(X)$ into the disjoint union

$$\mathcal{S}^{\mathrm{Tor}}_{d,k}(X) = \{K \in \mathcal{S}^{\mathrm{Tor}}_{d,k}(X) : D < c_{d,k}\} \sqcup \{K \in \mathcal{S}^{\mathrm{Tor}}_{d,k}(X) : D \ge c_{d,k}\} =: \mathcal{S}^{\mathrm{Tor}}_{d,k}(c_{d,k}) \sqcup \mathcal{S}^{\mathrm{Tor},1}_{d,k}(X).$$

Lemma 4.3. *We have* $\mathcal{S}_{d,k}^{\mathrm{Tor},1}(X) \subset \mathcal{S}_{d,k}^{\mathrm{NV}}(X)$.

Proof. Let $K \in \mathcal{S}_{d,k}^{\mathrm{Tor},1}(X)$ and define the cyclotomic extension $N := K(\zeta_{2k-1})$ where ζ_{2k-1} is a primitive $(2k-1)$-th root of unity. Recall that the Galois group $G_k := \mathrm{Gal}(\overline{\mathbb{Q}}/N)$ acts on $\Psi_{d,k}(D)$ by

$$\psi \mapsto \psi^\sigma := \sigma \circ \psi, \quad \sigma \in G_k.$$

For a fixed $\psi_0 \in \Psi_{d,k}(D)$, we denote the Galois orbit of ψ_0 by

$$\mathcal{O}_{\psi_0} = \{\psi_0^\sigma : \sigma \in G_k\}.$$

Now, since $D \geq c_{d,k}$, by Theorem 1.4 there exists a $\psi_0 \in \Psi_{d,k}(D)$ such that $L(\psi_0, k) \neq 0$. Also, using work of Shimura [Shi], one can show that for any $\sigma \in G_k$, we have

$$L(\psi_0^\sigma, k) \neq 0 \quad \text{if and only if} \quad L(\psi_0, k) \neq 0.$$

Hence it follows that

$$NV_{d,k}(D) \geq \#\mathcal{O}_{\psi_0}.$$

On the other hand, by [M, Proposition 1.1] we have

$$\#\mathcal{O}_{\psi_0} = \frac{h(-D)}{\#\mathrm{Cl}_{2k-1}(K)}.$$

Therefore

$$NV_{d,k}(D) \geq \frac{h(-D)}{\#\mathrm{Cl}_{2k-1}(K)}.$$

By Siegel's theorem, we have

$$h(-D) \gg_\epsilon D^{1/2-\epsilon}.$$

Then since K satisfies the bound (1.4) with $\ell = 2k-1$, we get

$$NV_{d,k}(D) \gg_\epsilon D^{\frac{1}{2(2k-1)}-\epsilon}.$$

It follows that $K \in \mathcal{S}_{d,k}^{\mathrm{NV}}(X)$. $\square$

Using Lemma 4.3, we get the decomposition

$$\mathcal{S}_{d,k}^{\mathrm{NV}}(X) = \mathcal{S}_{d,k}^{\mathrm{Tor},1}(X) \sqcup \left(\mathcal{S}_{d,k}^{\mathrm{NV}}(X) \setminus \mathcal{S}_{d,k}^{\mathrm{Tor},1}(X)\right).$$

Now, since

$$\#\mathcal{S}_{d,k}^{\mathrm{Tor}}(c_{d,k}) \ll_{d,k} 1,$$

by Proposition 4.1 we get

$$\begin{aligned}\#\mathcal{S}_{d,k}^{\mathrm{Tor},1}(X) &= \#\mathcal{S}_{d,k}^{\mathrm{Tor}}(X) - \#\mathcal{S}_{d,k}^{\mathrm{Tor}}(c_{d,k}) \\ &= 2^{-m}\left(1 - \prod_{i=1}^{n}\left(\frac{1}{1+q_i}\right)\right)\prod_{i=1}^{m}\left(\frac{1}{1+p_i^{-1}}\right)\frac{X}{6\zeta(2)} + O_{d,k}(X^{1-\frac{1}{2(2k-1)}}).\end{aligned}$$

Also, since

$$\mathcal{S}_{d,k}^{\mathrm{NV}}(X) \setminus \mathcal{S}_{d,k}^{\mathrm{Tor},1}(X) \subset \mathcal{S}_{d,k}^{\neg\mathrm{Tor}}(X),$$

the bound (4.1) gives

$$\#\left(\mathcal{S}_{d,k}^{\mathrm{NV}}(X) \setminus \mathcal{S}_{d,k}^{\mathrm{Tor},1}(X)\right) \leq \#\mathcal{S}_{d,k}^{\neg\mathrm{Tor}}(X) \ll X^{1-\frac{1}{2(2k-1)}}.$$

Hence

$$\#\mathcal{S}_{d,k}^{\mathrm{NV}}(X) = 2^{-m}\left(1-\prod_{i=1}^{n}\left(\frac{1}{1+q_i}\right)\right)\prod_{i=1}^{m}\left(\frac{1}{1+p_i^{-1}}\right)\frac{X}{6\zeta(2)} + O_{d,k}(X^{1-\frac{1}{2(2k-1)}}).$$

This proves Theorem 1.2.

Finally, by combining the preceding asymptotic formula with Proposition 4.1, we get

$$\frac{\#\mathcal{S}_{d,k}^{\mathrm{NV}}(X)}{\#\mathcal{S}_{d,k}(X)} = 1 + O_{d,k}(X^{-\frac{1}{2(2k-1)}}).$$

This proves Corollary 1.3. □

References

[BK] S. Bloch and K. Kato, *L–functions and Tamagawa numbers of motives.* The Grothendieck Festschrift I, P. Cartier, et al., eds., Prog. in Math. **86**, Boston: Birkhauser (1990) 333–400.

[BD] G. Boxer and P. Diao, *Effective nonvanishing of canonical Hecke L–functions.* Proc. Amer. Math. Soc. **138** (2010), 3891–3897.

[EV] J. Ellenberg and A. Venkatesh, *Reflection Principles and Bounds for Class Group Torsion.* Int. Math. Res. Not. no.1, Art ID rnm002 (2007).

[EPW] J. Ellenberg, L. B. Pierce, M. Matchett Wood, *On ℓ-torsion in Class Groups of Number Fields.* `arXiv:1606.06103 [math.NT]`.

[G] B. Gross, *Arithmetic on elliptic curves with complex multiplication. With an appendix by B. Mazur.* Lecture Notes in Mathematics, **776**. Springer, Berlin, 1980. iii+95 pp.

[KMY] B. D. Kim, R. Masri, and T. Yang, *Nonvanishing of Hecke L-Functions and the Bloch-Kato Conjecture*, Mathematische Annalen, vol. 349 (2011), 301-343.

[LX] C. Liu and L. Xu, *The vanishing order of certain Hecke L–functions of imaginary quadratic fields*, J. Number Theory, **108** (2004), 76–89.

[M2] R. Masri, *Asymptotics for sums of central values of canonical Hecke L–series.* Int. Math. Res. Not. IMRN 2007, no. 19, Art. ID rnm065, 27 pp.

[M] R. Masri, *Quantitative nonvanishing of L-series associated to canonical Hecke characters.* Int. Math. Res. Not. IMRN 2007, no. 19, Art. ID rnm070, 16 pp.

[MV] P. Michel and A. Venkatesh, *Heegner points and non-vanishing of Rankin/Selberg L-functions.* Analytic number theory, 169–183, Clay Math. Proc., **7**, Amer. Math. Soc., Providence, RI, 2007.

[MY] S. D. Miller and T. Yang, *Nonvanishing of the central derivative of canonical Hecke L–functions.* Math. Res. Lett. **7** (2000), 263–277.

[MR] H. Montgomery and D. Rohrlich, *On the L-functions of canonical Hecke characters of imaginary quadratic fields. II*, Duke Math. J. **49** (1982), 937–942.

[RVY] F. Rodriguez-Villegas and T. Yang, *Central values of Hecke L–functions of CM number fields.* Duke Math. J. **98** (1999), 541–564.

[R1] D. Rohrlich, *The nonvanishing of certain Hecke L-functions at the center of the critical strip.* Duke Math. J. **47** (1980), 223–232.

[R2] D. Rohrlich, *On the L-functions of canonical Hecke characters of imaginary quadratic fields.* Duke Math. J. **47** (1980), 547–557.

[R3] D. Rohrlich, *Galois conjugacy of unramified twists of Hecke characters.* Duke Math. J. **47** (1980), 695–703.

[R4] D. Rohrlich, *Root numbers.* Arithmetic of L–functions, 353–448, IAS/Park City Math. Ser., **18**, Amer. Math. Soc., Providence, RI, 2011.

[Shi] G. Shimura, *The special values of the zeta functions associated to cusp forms*, Comm. Pure Appl. Math. 29 (1976), 783-804.

[Y] T. Yang, *Nonvanishing of central Hecke L-values and rank of certain elliptic curves.* Compositio Math. **117** (1999), 337–359.