ZEROS OF NEWFORM EISENSTEIN SERIES ON $\Gamma_0(N)$

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ABSTRACT. We examine the newform Eisenstein series $E_{\chi_1,\chi_2,k}(z)$ of weight k on $\Gamma_0(q_1 q_2)$, where χ_1 and χ_2 are primitive characters modulo q_1 and q_2 respectively. We determine the location and distribution of a significant fraction of the zeros of these Eisenstein series for k sufficiently large.

1. INTRODUCTION

1.1. **Statement of Results.** The zeros of the classical Eisenstein series E_k for $k \geq 4$ were studied in [RS-D], where it was shown that when E_k is restricted to the standard fundamental domain $\mathcal{F}$, its zeros rest entirely on the boundary $|z| = 1$. By contrast, the zeros of weight k Hecke cusp forms equidistribute in $\mathcal{F}$ by [R] and [HS]. For cusp forms as the level tends to infinity, although Quantum Unique Ergodicity is known (see [N], [NPS]), the corresponding equidistribution of zeros is unknown. In this paper we study the zeros of newform Eisenstein series where both the weight and level may vary.

Let χ_1 and χ_2 be primitive characters modulo q_1 and q_2, respectively, with $q_1, q_2 > 1$. We consider newform Eisenstein series with nebentypus $\chi_1 \overline{\chi_2}$ on the congruence subgroup $\Gamma_0(q_1 q_2)$ of $\mathrm{SL}_2(\mathbb{Z})$. These Eisenstein series are defined by

$$E_{\chi_1,\chi_2,k}(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{\chi_1(c)\chi_2(d)}{(cq_2 z + d)^k}, \tag{1}$$
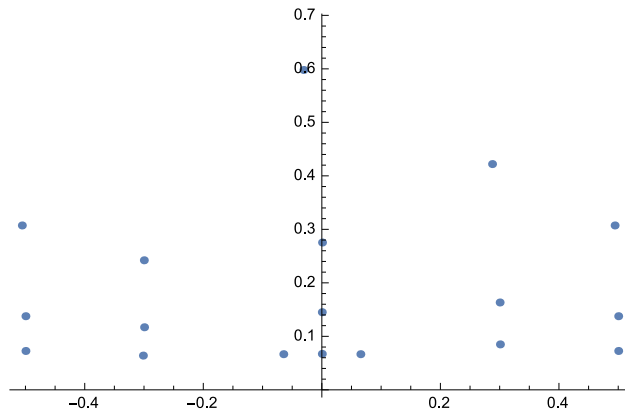
where, in order to avoid triviality, we assume that

$$\chi_1(-1)\chi_2(-1) = (-1)^k, \qquad k \geq 3.$$

These series have a Fourier expansion given in [DS, Theorem 4.5.1] as

$$E_{\chi_1,\chi_2,k}(z) = e(\chi_1, \chi_2, k) \sum_{n=1}^{\infty} \left(\sum_{ab=n} \chi_1(a)\overline{\chi_2}(b) b^{k-1} \right) e(nz), \tag{2}$$

where $e(\chi_1, \chi_2, k)$ is some constant independent of z whose value does not affect the location of zeros.

In this paper, we determine the location of a distinguished subset of zeros of $E_{\chi_1,\chi_2,k}(z)$.

FIGURE 1. An example for $q_1 = 3$, $q_2 = 5$, $k = 10$.

We may see a specific example of the vanishing of $E_{\chi_1,\chi_2,k}(x + iy)$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and $\frac{1}{2\sqrt{3}q_2} \ll y \ll \frac{\sqrt{k}}{q_2}$. In Figure 1, we have that χ_1 is the Legendre symbol modulo $q_1 = 3$, χ_2 is the unique character modulo 5 such that $\chi_2(2) = i$, and $k = 8$. In this figure and in other computations, we notice some vertical lines of zeros, which motivates the following theorem.

Theorem 1.1. *Let $a \in \mathbb{Z}$ be such that $\gcd(a, q_2) = \gcd(a + 1, q_2) = 1$. Then $E_{\chi_1,\chi_2,k}(z)$ has m zeros which are each within $O\left(\frac{1}{q_2 k}\right)$ of the line $x = \frac{a + 1/2}{q_2}$. We have that m satisfies*

$$m = \frac{k}{3} - O(\sqrt{k}),$$

with an absolute implied constant.

With extra work, one could derive explicit constants for the above theorem. We note that once k is sufficiently large, then one is free to vary q_1 and q_2 and the results are uniform in these parameters.

Additionally, we will demonstrate that the zeros found in Theorem 1.1, for a fixed integer a, are equidistributed with respect to a certain angle θ defined in (3) as k tends to infinity. Furthermore, we see in Section 3.5 that if $q_1 > 3$, these zeros are $\Gamma_0(q_1 q_2)$-inequivalent.

Theorem 1.1 is approached using the $cz + d$ expansion from (1) for $\frac{1}{2\sqrt{3}q_2} \ll \text{Im}(z) \ll \frac{\sqrt{k}}{q_2}$.

In a complementary range where $\text{Im}(z) \gg \sqrt{k}$, we use the Fourier expansion to approximate $E_{\chi_1,\chi_2,k}(z)$, which is motivated by the ideas of [GS]. Taking the $n = \ell$ and $n = \ell + 1$ terms of the Fourier expansion gives a good approximation to $E_{\chi_1,\chi_2,k}(z)$ for $y = \text{Im}(z)$ in the following range:

$$\frac{k - 1}{2\pi(\ell + 1)} =: y_{\ell+1} \leq y \leq y_\ell := \frac{k - 1}{2\pi\ell}.$$

Theorem 1.2. *Let ℓ be a natural number with $(\ell, q_2) = (\ell + 1, q_2) = 1$ and $\ell \leq \epsilon \sqrt{k}$ for a sufficiently small $\epsilon > 0$ and sufficiently large k. Then $E_{\chi_1, \chi_2, k}(z)$ has exactly one zero for $-\frac{1}{2} < x \leq \frac{1}{2}$ and $y_{\ell+1} \leq y \leq y_\ell$.*

We note that this result is also uniform in q_1, q_2, and k.

Due to the constraints of these expansions, we are unable to locate zeros where $\frac{\sqrt{k}}{q_2} \ll y \ll \sqrt{k}$. Note that Theorem 1.1 provides the location of roughly $\gg \varphi(q_2)k$ zeros and Theorem 1.2 provides roughly $\sqrt{k}$ zeros. These zeros are produced in a neighborhood around infinity. Additionally, in Section 5 we study $E_{\chi_1, \chi_2, k}(z)$ near Atkin-Lehner cusps in order to find additional zeros.

1.2. **Heuristic Discussion on Equidistribution.** One of our primary motivations for this work is gathering evidence as to whether the zeros of newform Eisenstein series equidistribute. A natural way to define equidistribution of a discrete set of points in $\Gamma_0(N)\backslash\mathbb{H}$ is if the points equidistribute in $\Gamma_0(1)\backslash\mathbb{H}$ after application of the projection map

$$\pi : \Gamma_0(N)\backslash\mathbb{H} \to \Gamma_0(1)\backslash\mathbb{H}.$$

As an example, Figure 2 shows the image of the zeros from Figure 1 under the map π.
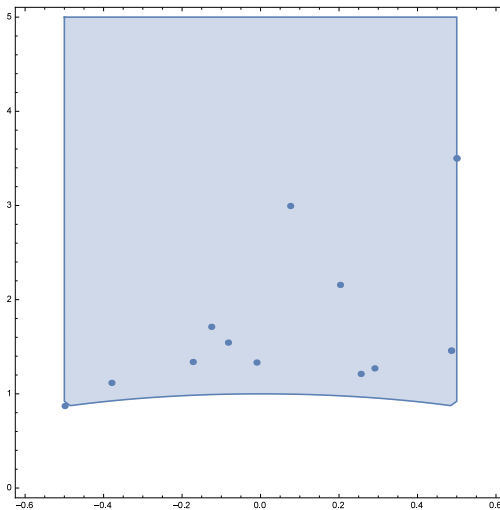

FIGURE 2. $q_1 = 3$, $q_2 = 7$, $k = 8$

We note the stark contrast between these zeros, which lie on the interior of $\mathcal{F}$, and the zeros found in [RS-D], which lie entirely on the bottom arc $|z| = 1$. Taking q_2 to be an odd prime, we will see in Section 3 that we have $q_2 - 2$ vertical lines of zeros of $E_{\chi_1, \chi_2, k}(z)$ with real part strictly between 0 and 1. Consider the lowest zero from each of these lines, which is within $O\left(\frac{1}{q_2 k}\right)$ of $\omega_a := \frac{a + 1/2}{q_2} + \frac{i}{2\sqrt{3}q_2}$ by Theorem 1.1. We claim that $\Gamma_0(1)\{\omega_a : 1 \leq a \leq q_2 - 2\}$ is a subset of the set of Hecke points, $T_p(z)$, for $z = \frac{1}{2} + \frac{i}{2\sqrt{3}}$, and $p = q_2$. For $z \in \mathbb{H}$ and

p a prime, the set of Hecke points is defined as the $\Gamma_0(1)$-orbits of the points in the set $\{\frac{z\pm a}{p} : a \pmod{p}\} \cup \{pz\}$. Note that the set of ω_a's is missing only three points from $T_p(z)$.

It is known that Hecke points $T_p(z)$ equidistribute in $\Gamma_0(1)\backslash\mathbb{H}$ as $p \to \infty$ for z fixed; we refer to [MV] Section 1.2 for discussion on the necessary equidistribution results. If we imagine our low-lying zeros are modeled by a random perturbation of Hecke points, it is reasonable to believe that they equidistribute in $\Gamma_0(1)\backslash\mathbb{H}$ as $q_2 \to \infty$. This discussion gives some heuristic evidence for why the zeros displayed in Figure 2 hint at equidistribution.

2. Outline of the Approach

In order to determine where the zeros of $E_{\chi_1,\chi_2,k}(z)$ lie, we will first distinguish small regions where the Eisenstein series is well approximated by very few terms. In Section 3, we will look at two terms from the $cz + d$ expansion, evaluated on a thin vertical strip. In Section 4, we will look at two terms from the Fourier expansion, which are evaluated in a horizontal strip for $y_{\ell+1} \leq y \leq y_\ell$.

These regions will be selected so that our main term is sufficiently large along its boundary, and the main term contains one zero within the region. We will then use Rouché's Theorem to demonstrate that the Eisenstein series also has a zero within the designated region. This theorem is restated for the reader's convenience.

Theorem 2.1 (Rouché). *Let E and g be two complex-valued functions which are holomorphic on a closed region K with rectangular boundary ∂K. If the strict inequality holds:*

$$|E(z) - g(z)| < |E(z)| + |g(z)|,$$

for all $z \in \partial K$, then E and g have the same number of zeros (including multiplicity) in the interior of K.

3. Zeros in the Region $\frac{1}{2\sqrt{3}} \ll q_2 \mathrm{Im}(z) \ll \sqrt{k}$

3.1. **The Main Terms of $E_{\chi_1,\chi_2,k}(z)$ Along the Line** $x = \frac{a+1/2}{q_2}$**.** We fix an integer a such that $\gcd(a, q_2) = \gcd(a + 1, q_2) = 1$. In a small region around the line $x = \frac{a+1/2}{q_2}$, we will see that the terms in the $cz + d$ expansion where $c = 1$ and $d = -a, -a - 1$ are a good approximation to $E_{\chi_1,\chi_2,k}(z)$. We denote these terms as

$$g_a(z) := \frac{\chi_2(-a)}{(q_2 z - a)^k} + \frac{\chi_2(-a-1)}{(q_2 z - a - 1)^k}.$$

We first wish to determine where our main term $g_a(z)$ has roots in a small region around the vertical line $x = \frac{a+\frac{1}{2}}{q_2}$. We first observe that, in order to have $g_a(z) = 0$, we must have

that the magnitudes of the two terms are equal. That is,

$$\frac{1}{|q_2 z - a|^k} = \frac{1}{|q_2 z - a - 1|^k},$$

which implies that $x = \frac{a+1/2}{q_2}$. Along this vertical line, we make a substitution to polar coordinates given by

$$z = \frac{a + 1/2}{q_2} + iy = \frac{a}{q_2} + \frac{R}{q_2} e^{i\theta}. \tag{3}$$

These coordinates will be used frequently and, for the reader's convenience, are illustrated in Figure 3. With this substitution, we see that

$$q_2 z - a = \frac{1}{2} + iq_2 y = Re^{i\theta}$$

$$q_2 z - a - 1 = -\frac{1}{2} + iq_2 y = -Re^{-i\theta}.$$

Therefore along $x = \frac{a+1/2}{q_2}$, we may see that $g_a(z) = 0$ reduces to

$$0 = \frac{\chi_2(-a)}{(q_2 z - a)^k} + \frac{\chi_2(-a-1)}{(q_2 z - a - 1)^k} = \frac{\chi_2(a)e^{-i\theta k} + (-1)^k \chi_2(a+1)e^{i\theta k}}{R^k}.$$

This is satisfied if and only if

$$e^{2i\theta k} + (-1)^k \chi_2(a)\overline{\chi_2(a+1)} = 0. \tag{4}$$

We conclude that the zeros of $g_a(z)$ along the vertical line $x = \frac{a+1/2}{q_2}$ depend only on their angle from the ray emerging from $\frac{a}{q_2}$. It now suffices to show that as k is sufficiently large, both $g_a(z)$ and $E_{\chi_1,\chi_2,k}(z)$ have the same number of zeros in small regions surrounding each zero of $g_a(z)$ along this vertical line.

3.2. **Defining the Regions Containing Zeros.** We fix a positive ϵ which is sufficiently small, and independent of q_1, q_2, and k. Let K_ϵ denote a small region around our vertical line:

$$K_\epsilon = \left\{ z = x + iy : \frac{a + \frac{1}{2} - \frac{\epsilon}{k}}{q_2} \le x \le \frac{a + \frac{1}{2} + \frac{\epsilon}{k}}{q_2}, \ \frac{1}{2\sqrt{3}q_2} < y < \frac{c\sqrt{k}}{q_2} \right\},$$

where c is some small absolute constant. A view of K_ϵ is displayed as the shaded region in Figure 3.

We will prove that the conditions hold to apply Rouché's Theorem for $g_a(z)$ and $E_{\chi_1,\chi_2,k}(z)$ on the boundaries of a series of regions $W_n \subset K_\epsilon$. Each region W_n will be chosen such that it contains exactly one zero of $g_a(z)$, and $|g_a(z)|$ is large on ∂W_n.

We pick θ_1 to be the smallest positive real number such that the following conditions hold

$$|e^{2ik\theta_1} + (-1)^k \chi_2(a)\overline{\chi_2(a+1)}| = 2 \tag{5}$$
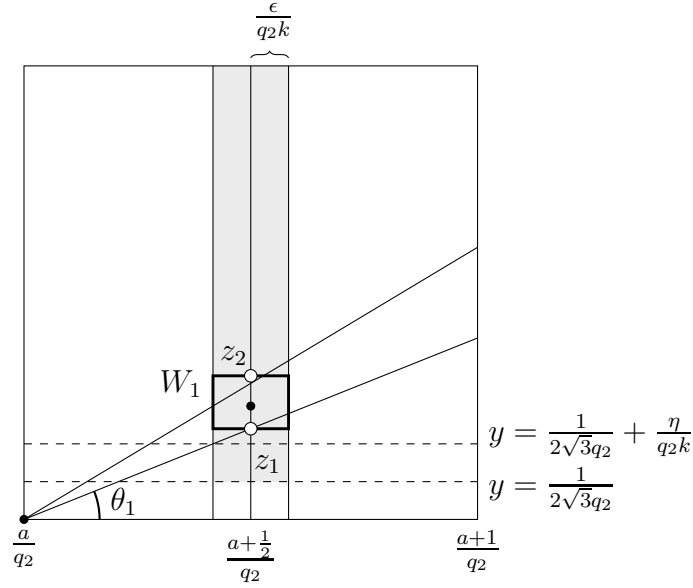
FIGURE 3. Coordinates around $x = \frac{a+1/2}{q_2}$

and

$$\tan(\theta_1) > \frac{1}{\sqrt{3}} + \frac{2\eta}{k} \tag{6}$$

for $\eta > 0$ large enough (this constant is motivated in the proof of Corollary 3.5). Our condition in equation (6) is equivalent to stating that the point at $z_1 := \frac{a}{q_2} + r_1 e^{i\theta_1}$ with real part $\mathrm{Re}(z) = \frac{a+1/2}{q_2}$ satisfies $\mathrm{Im}(z) > \frac{1}{2\sqrt{3}q_2} + \frac{\eta}{q_2 k}$. This guarantees that it lies sufficiently far from the lower boundary of K_ϵ.

Note also that if θ_1 satisfies (5), then $\theta_{n+1} := \theta_1 + \frac{n\pi}{k}$ also satisfies (5) for each natural number n. With this in mind, we define z_n for $n \geq 1$ to satisfy

$$z_n = \frac{a}{q_2} + r_n e^{i(\theta_1 + \frac{n\pi}{k})}, \qquad \mathrm{Re}(z_n) = \frac{a+1/2}{q_2}.$$

One may additionally verify that

$$|g_a(z_n)| = \frac{2}{r_n^k} \tag{7}$$

where $r_n = \sqrt{\frac{1}{4} + q_2^2 \cdot \mathrm{Im}(z_n)^2}$.

Note that z_1 and z_2 are illustrated in Figure 3 as the two open points. We finally define W_n to be the rectangle in K_ϵ given by

$$W_n = \{z = x + iy \in K_\epsilon : \mathrm{Im}(z_n) \leq y \leq \mathrm{Im}(z_{n+1})\}.$$

Additionally, by our definition of θ_1, we have that W_1 is the lowest-lying region of this form to lie within $\frac{\eta}{k}$ inside of K_ϵ. Its boundary is illustrated as the bolded rectangle in Figure 3.

We also see that in Figure 3, the black dot inside of W_1 is the unique zero of $g_a(z)$ inside of W_1. In general, we may define $\theta'_n = \frac{\theta_n + \theta_{n+1}}{2} = \theta_n + \frac{\pi}{2k}$, and see that this is the unique argument in equation (4) which yields a zero of $g_a(z)$ inside W_n. This discussion is summarized in the following Proposition:

Proposition 3.1. *The main term $g_a(z)$ has exactly one zero in each region W_n occurring at the argument θ'_n in the coordinates defined in equation (3).*

We now determine the number of zeros of $g_a(z)$ in K_ϵ.

3.3. **Bounds on the Number of Zeros of $g_a(z)$ in K_ϵ.** It suffices to determine the highest W_n which is contained within K_ϵ. We define m to be the largest integer satisfying

$$\frac{1}{2q_2} \tan\left(\theta_1 + \frac{m\pi}{k}\right) < \frac{c\sqrt{k}}{q_2}.$$

for some sufficiently small constant c. That is, the imaginary part of the point z_m is less than our upper height on K_ϵ. Note that $\theta_1 + \frac{m}{k}\pi \leq \arctan\left(2c\sqrt{k}\right)$, so

$$m = \frac{k}{\pi}\left(\arctan\left(2c\sqrt{k}\right) - \arctan\left(\frac{1}{\sqrt{3}} + \frac{2\eta}{k}\right)\right) + O(1)$$

$$= \frac{k}{3} - O(\sqrt{k}).$$

This will lead us to prove that the conditions for Rouché's Theorem hold for each W_n where $1 \leq n \leq m - 1$.

3.4. **Inequalities on ∂W_n.** Before stating our lemmas, we first include a brief proposition which will be useful for deriving lower bounds for $g_a(z)$.

Proposition 3.2. *For a complex number Y with $|Y| > 1$ and δ complex such that $|\delta|$ is sufficiently small, we have that*

$$\left|Y + \frac{\delta}{k}\right|^k = |Y|^k\left(1 + O\left(\left|\frac{\delta}{Y}\right|\right)\right).$$

Proof.

$$\left|Y + \frac{\delta}{k}\right|^k = \exp\left(k\log\left(\left|Y + \frac{\delta}{k}\right|\right)\right) = \exp\left(k\log(|Y|) + kO\left(\frac{|\delta|}{|Y|k}\right)\right)$$

$$= |Y|^k| \cdot \exp\left(O\left(\left|\frac{\delta}{Y}\right|\right)\right) = |Y|^k\left(1 + O\left(\left|\frac{\delta}{Y}\right|\right)\right). \qquad \blacksquare$$

We will now prove the following Lemmas.

Lemma 3.3. *In K_ϵ, we have that, with an absolute implied constant,*

$$\left| E_{\chi_1,\chi_2,k}(z) - g_a(z) \right| \ll (2q_2 y)^{-k} + q_2 y \left(\frac{9}{4} + q_2^2 y^2 \right)^{-k/2}. \tag{8}$$

Lemma 3.4. *The following bound holds for $g_a(z)$ on the boundary ∂W_n, where $1 \leq n \leq m-1$:*

$$|g_a(z)| \gg \left(\frac{1}{4} + q_2^2 y^2 \right)^{-k/2-1}. \tag{9}$$

Lemma 3.3 follows directly from the proofs of Lemmas 3.2 and 3.4 in [RVY]. We will prove Lemma 3.4 below, but first we note that by these lemmas, we obtain the following corollary as a result:

Corollary 3.5. *For sufficiently large k, we have that $E_{\chi_1,\chi_2,k}(z)$ has a unique zero in W_n for each $1 \leq n \leq m-1$.*

Proof of Corollary 3.5. On the boundary ∂W_n, where $1 \leq n \leq m-1$, we claim that the following inequality holds

$$|g_a(z)| > |E_{\chi_1,\chi_2,k}(z) - g_a(z)|.$$

Using Lemmas 3.3 and 3.4 on ∂W_n, we now argue that the bounds in (9) are significantly larger than those in (8), so long as $q_2 y$ is within a determined range. We have that

$$q_2 y \left(\frac{9}{4} + q_2^2 y^2 \right)^{-k/2} = o\left(\left(\frac{1}{4} + q_2^2 y^2 \right)^{-k/2-1} \right),$$

provided that

$$q_2 y = o(\sqrt{k}).$$

Additionally, we have that

$$(2q_2 y)^{-k} = o\left(\left(\frac{1}{4} + q_2^2 y^2 \right)^{-k/2-1} \right),$$

so long as $q_2 y \geq 1$, as k tends to infinity. We may also see that

$$\frac{(2q_2 y)^{-k}}{\left(\frac{1}{4} + q_2^2 y^2 \right)^{-k/2-1}}$$

becomes sufficiently small when

$$q_2 y > \frac{1}{2\sqrt{3}} + \frac{\eta}{k},$$

for an absolute constant η which is large enough compared to the implied constants in (8) and (9).

On ∂W_n, this gives us that $|E_{\chi_1,\chi_2,k} - g_a(z)|$ vanishes quicker than $|g_a(z)|$ as k tends to infinity. In particular, for a sufficiently large k, we have that

$$|g_a(z)| > |E_{\chi_1,\chi_2,k}(z) - g_a(z)|.$$

Then by Rouché's Theorem, we obtain that $g_a(z)$ and $E_{\chi_1,\chi_2,k}(z)$ have the same number of zeros in W_n for each $1 \leq n \leq m - 1$. By Proposition 3.1, the result follows. ∎

We note that Theorem 1.1 follows from the previous lemmas and corollary. It only remains to prove Lemma 3.4.

Proof of Lemma 3.4. First we look at the right boundary of K_ϵ, where $\text{Re}(z) = \frac{a + \frac{1}{2} + \frac{\epsilon}{k}}{q_2}$. By the reverse triangle inequality on this vertical line segment,

$$|g_a(z)| \geq \frac{1}{\left(\left(\frac{1}{2} + \frac{\epsilon}{k}\right)^2 + q_2^2 y^2\right)^{k/2}} \left(\left(\frac{\left(\frac{1}{2} + \frac{\epsilon}{k}\right)^2 + q_2^2 y^2}{\left(\frac{1}{2} - \frac{\epsilon}{k}\right)^2 + q_2^2 y^2}\right)^{k/2} - 1\right).$$

Expanding the term on the right, we obtain

$$
\begin{aligned}
\left(\frac{\left(\frac{1}{2} + \frac{\epsilon}{k}\right)^2 + q_2^2 y^2}{\left(\frac{1}{2} - \frac{\epsilon}{k}\right)^2 + q_2^2 y^2}\right)^{k/2} - 1 &= \left(1 + \frac{2\frac{\epsilon}{k}}{\left(\frac{1}{2} - \frac{\epsilon}{k}\right)^2 + q_2^2 y^2}\right)^{k/2} - 1 \\
&\geq \exp\left(\frac{k}{2} \log\left(1 + \frac{2\epsilon}{k\left(\frac{1}{4} + q_2^2 y^2\right)}\right)\right) - 1 \\
&= \exp\left(\frac{\epsilon}{\left(\frac{1}{4} + q_2^2 y^2\right)} + O\left(\frac{\epsilon^2}{k\left(\frac{1}{4} + q_2^2 y^2\right)^2}\right)\right) - 1 \gg \frac{\epsilon}{\left(\frac{1}{4} + q_2^2 y^2\right)}.
\end{aligned}
$$

This gives us that on the right boundary of K_ϵ,

$$|g_a(z)| \gg_\epsilon \frac{1}{\left|\frac{1}{4} + q_2^2 y^2\right|^{k/2+1}}$$

by Proposition 3.2.

A symmetric argument holds when $\text{Re}(z) = \frac{a + \frac{1}{2} - \frac{\epsilon}{k}}{q_2}$. We now turn our attention to the bottom segment of ∂W_n.

Recall from equation (7), we have that

$$|g_a(z_n)| = \frac{2}{\left(\frac{1}{4} + q_2^2 y^2\right)^{k/2}}$$

Letting δ vary from $0 \leq \delta \leq \epsilon$, we have $z = z_n + \frac{\delta}{q_2 k}$ on the lower boundary of ∂W_n. We may use Proposition 3.2 to show

$$\frac{\chi_2(-a)}{(\frac{1}{2} + \frac{\delta}{k} + iq_2y)^k} = \frac{\chi_2(-a)}{(\frac{1}{2} + iq_2y)^k \left(1 + \frac{\delta}{k(\frac{1}{2}+iq_2y)}\right)^k} = \frac{\chi_2(-a)}{(\frac{1}{2} + iq_2y)^k}\left(1 + O\left(\frac{\epsilon}{|\frac{1}{2} + iq_2y|}\right)\right).$$

We use this to see that

$$|g_a(z)| = |g_a(z_n)|\left(1 + O\left(\frac{\epsilon}{|\frac{1}{2} + iq_2y|}\right)\right) \gg_\epsilon \left|\frac{1}{4} + q_2^2 y^2\right|^{-k/2}.$$

∎

3.5. On the $\Gamma_0(q_1 q_2)$-inequivalence of Zeros.

Proposition 3.6. *The points contained in the region with real part $-\frac{1}{2} < x \le \frac{1}{2}$ and imaginary part $Im(z) > \frac{1}{q_1 q_2}$ are $\Gamma_0(q_1 q_2)$-inequivalent.*

Proof. Suppose we have two points z and z' in K_ϵ such that $Im(z') \ge Im(z)$, and a $\gamma \in \Gamma_0(q_1 q_2)$ such that $\gamma z = z'$. Then we must have that

$$|cz + d|^2 \le 1$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We then have that $c^2 y^2 \le 1$, which implies that $|c| < q_1 q_2$ by our lower bound on $y = Im(z)$ in K_ϵ. Since $c \equiv 0 \pmod{q_1 q_2}$ we must have that $c = 0$ and $d = \pm 1$, that is, γ is a translate. ∎

Corollary 3.7. *If $q_1 > 3$, all the zeros found in Theorem 1.1 are $\Gamma_0(q_1 q_2)$-inequivalent.*

3.6. A Remark On $(a+1, q_2) > 1$.
In the case where a and $a+1$ are not both coprime to q_2, we may carry out a similar argument. Let $(a, q_2) = (a+b, q_2) = 1$ such that $(a+t, q_2) > 1$ for all integers $0 < t < b$. We then obtain zeros approaching the line $x = \frac{a+b/2}{q_2}$ given by

$$z = \frac{a + b/2}{q_2} + iy = \frac{a}{q_2} + Re^{i\theta}$$

for θ satisfying

$$|e^{2i\theta k} + (-1)^k \chi_2(a)\overline{\chi_2(a+b)}| = 0.$$

We can illustrate this in the special case of finding zeros around the imaginary axis in $\mathbb{H}$.

In a neighborhood around $x = 0$, we have that the main terms of $E_{\chi_1, \chi_2, k}(x + iy)$ are

$$g(z) = \frac{\chi_2(-1)}{(q_2 z - 1)^k} + \frac{\chi_2(1)}{(q_2 z + 1)^k}.$$

We have the same upper bounds on $|E_{\chi_1,\chi_2,k}(z)-g(z)|$ from the above theorem and analogous lower bounds for $|g(z)|$ on the boundary. We then obtain zeros approaching the line $x = 0$ as k tends to infinity, when

$$|e^{2i\theta k} + (-1)^k\chi_2(-1)| = 0.$$

However we note that $(-1)^k\chi_2(-1) = \chi_1(-1)$. This gives us that the zeros of the main term are of the form

$$\frac{-1}{q_2} + Re^{i\theta} = iy.$$

Therefore these zeros depend only on the parameters $\operatorname{sgn}(\chi_1)$ and q_2.

4. Zeros in the Region $\operatorname{Im}(z) \gg \sqrt{k}$

4.1. **The Fourier Expansion.** For this portion of the paper, let

$$F(z) = \sum_{n=1}^{\infty}\left[\sum_{ab=n}\chi_1(a)\overline{\chi_2}(b)b^{k-1}\right]e(nz).$$

Recalling (2), the zeros of $F(z)$ are the zeros of $E_{\chi_1,\chi_2,k}(z)$. In the definition of $F(z)$, we let

$$f_n(z) = \overline{\chi_2}(n)n^{k-1}e(nz) \tag{10}$$

and take $a = 1$ and $b = n$ to simplify the expansion to

$$F(z) = \sum_{n=1}^{\infty}f_n(z) + \delta(z),$$

where

$$|\delta(x+iy)| \leq \sum_{n=1}^{\infty}n^{k-1}\exp(-2\pi ny)\left(\sum_{\substack{b|n\\b<n}}\left(\frac{b}{n}\right)^{k-1}\right). \tag{11}$$

4.2. **The Main Term of $F(z)$.** Now, we define our main term, denoted $h_\ell(z)$. For our purposes, assume $\ell \leq \epsilon\sqrt{k}$ for a sufficiently small $\epsilon > 0$, and assume also that ℓ and $\ell+1$ are coprime to q_2. Then, we consider two terms of the Fourier expansion, $n = \ell$ and $n = \ell + 1$, and we define $h_\ell(z)$ to be the main term of the expansion:

$$h_\ell(z) = f_\ell(z) + f_{\ell+1}(z),$$

where we take z to be restricted to the region:

$$\frac{k-1}{2\pi(\ell+1)} =: y_{\ell+1} \leq y \leq y_\ell := \frac{k-1}{2\pi\ell}. \tag{12}$$

Write

$$F(z) = h_\ell(z) + \beta(z), \tag{13}$$

where $\beta(z) = f_{\ell+2}(z) + f_{\ell+3}(z) + f_{\ell-1}(z) + f_{\ell-2}(z) + \varepsilon_1(z) + \varepsilon_2(z) + \delta(z)$ and where $\varepsilon_1(z) = \sum_{n=1}^{\ell-3}f_n(z)$ and $\varepsilon_2(z) = \sum_{n=\ell+4}^{\infty}f_n(z)$.

Now, we find zeros of $h_\ell(z)$ in the region from (12):

Lemma 4.1. *The main term $h_\ell(z)$ has a unique zero $x_0 + iy_0$ in the region $-\frac{1}{2} < x \leq \frac{1}{2}$ and $y_{\ell+1} \leq y \leq y_\ell$, with x_0 and y_0 given by (14) and (15) below.*

Proof of Lemma 4.1. Setting $h_\ell(z) = 0$, we find that

$$-\overline{\chi_2}(\ell)\chi_2(\ell+1)\left(1 - \frac{1}{\ell+1}\right)^{k-1} = \exp(2\pi i x)\exp(-2\pi y).$$

Then,

$$-\overline{\chi_2}(\ell)\chi_2(\ell+1) = \exp(2\pi i x) \quad \text{and} \quad \left(1 - \frac{1}{\ell+1}\right)^{k-1} = \exp(-2\pi y).$$

Consequently, $x_0 \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ is the unique solution to

$$e(x_0) = -\overline{\chi_2}(\ell)\chi_2(\ell+1) \tag{14}$$

and

$$y_0 = -\frac{(k-1)}{2\pi}\log\left(1 - \frac{1}{\ell+1}\right) = \frac{(k-1)}{2\pi}\left|\log\left(1 - \frac{1}{\ell+1}\right)\right|. \tag{15}$$

Using

$$-\frac{1}{\ell} \leq \log\left(1 - \frac{1}{\ell+1}\right) \leq -\frac{1}{\ell+1},$$

we see that $y_0 \in (y_{\ell+1}, y_\ell)$, as desired. ∎

4.3. **Method to Prove Theorem 4.2.** We define

$$N(y, k) = \frac{(2\pi y)^k}{\Gamma(k)}$$

to be a natural normalization factor of $F(z)$. Make note that multiplying $F(z)$ by $N(y, k)$ will not affect the zeros of $F(z)$. Now, we explain the use of Rouché's Theorem in the context of this section. We must show that the strict inequality

$$N(y, k)\,|\beta(z)| = N(y, k)\,|F(z) - h_\ell(z)| < N(y, k)\,|F(z)| + N(y, k)\,|h_\ell(z)| \tag{16}$$

holds in the region V_ℓ, where

$$V_\ell = \left\{z = x + iy : x_0 - \frac{1}{2} \leq x \leq x_0 + \frac{1}{2},\ y_{\ell+1} \leq y \leq y_\ell\right\}.$$

If (16) holds, then $F(z)$ will have the same number of zeros as $h_\ell(z)$ in the region V_ℓ. On ∂V_ℓ, we will show

$$N(y, k)\,|\beta(z)| < N(y, k)\,|h_\ell(z)|, \tag{17}$$

which implies (16). This leads us to our main theorem, which is the same as Theorem 1.2.

Theorem 4.2. *The function $E_{\chi_1, \chi_2, k}(z)$ has exactly one zero in the region V_ℓ.*

To prove Theorem 4.2, we need the following:

Lemma 4.3. *Let $0 < t \leq \frac{1}{2}$, and define*

$$M(t) = \log(t) - \log(|\log(1-t)|) - \frac{\log(1-t) + t}{t} \tag{18}$$

$$R(t) = t - \log(1+t). \tag{19}$$

Then,

$$0 < M(t) < R(t) < t^2.$$

We include the graph of $M(t)$ and $R(t)$ in Figure 4 for the reader's convenience.
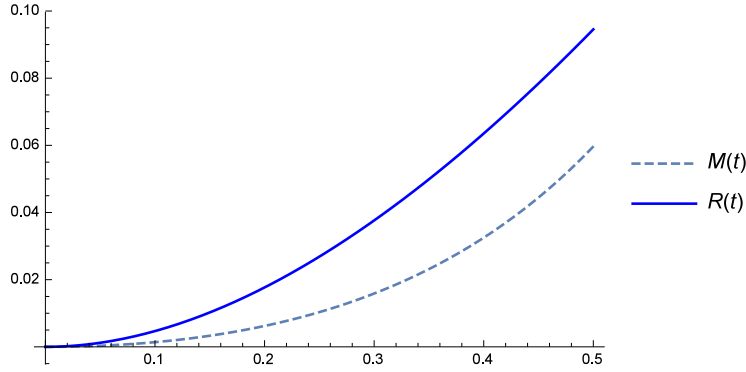

FIGURE 4. The Functions $M(t)$ and $R(t)$

Applying Lemma 4.3, we will show:

Lemma 4.4. *On ∂V_ℓ,*

$$N(y, k) |h_\ell(z)| \gg \frac{\sqrt{k}}{\ell} \exp\left(-k \cdot M\left(\frac{1}{\ell+1}\right)\right). \tag{20}$$

Lemma 4.5. *For all $z \in V_\ell$,*

$$N(y, k) |\beta(z)| \ll \frac{\sqrt{k}}{2^k \ell} + \frac{\sqrt{k}}{\ell} \exp\left(-k \cdot R\left(\frac{1}{\ell+1}\right)\right). \tag{21}$$

Theorem 4.2 will follow from Lemmas 4.1, 4.3, 4.4, and 4.5.

4.4. **Proofs of Lemmas 4.4 and 4.5.** Before beginning the proofs of Lemma 4.4 and 4.5, we must provide two facts that will aid us. If $0 < u < 1$, then

$$u - \frac{u^2}{2} \leq \log(1+u) \leq u - \frac{u^2}{4}. \tag{22}$$

Additionally, recall Stirling's approximation: For $n \in \mathbb{Z}^+$,

$$\Gamma(n) \sim \sqrt{\frac{2\pi}{n}} \left(\frac{n}{e}\right)^n. \tag{23}$$

Proof of Lemma 4.4. Consider two cases:

Top and Bottom Boundaries: Let $y = y_\ell$. Note that, from the definition of $h_\ell(z)$ and the triangle inequality,

$$N(y,k)\,|h_\ell(z)| \geq \frac{(2\pi y\ell)^k \exp(-2\pi\ell y)}{\Gamma(k)} \cdot \left|1 - \left|\frac{f_{\ell+1}(z)}{f_\ell(z)}\right|\right|.$$

In the next steps, we substitute $k - 1 = 2\pi y\ell$ and apply (23), which implies

$$\frac{e}{\ell}\left(\frac{k-1}{e}\right)^k \frac{1}{\Gamma(k)} \gg \frac{\sqrt{k}}{\ell}.$$

Furthermore, we get an upper bound on $\left|\frac{f_{\ell+1}(z)}{f_\ell(z)}\right|$ at $y = y_\ell$ using (22), namely

$$\left|\frac{f_{\ell+1}(z)}{f_\ell(z)}\right| = \left(1 + \frac{1}{\ell}\right)^{k-1} \exp\left(-\frac{k-1}{\ell}\right) \ll \exp\left(-\frac{k}{4\ell^2}\right). \tag{24}$$

Recall that $\ell \leq \epsilon\sqrt{k}$ for a sufficiently small $\epsilon > 0$, so (24) is less than $\frac{1}{2}$. Then,

$$N(y,k)|h_\ell(z)| \gg \frac{\sqrt{k}}{\ell} \tag{25}$$

for $y = y_\ell$. Letting $y = y_{\ell+1}$, by similar methods, we conclude that (25) holds.

Left and Right Boundaries: Let $x = x_0 \pm \frac{1}{2}$. Then, let $r_1(y)$ and $r_2(y)$ be the magnitudes of the two terms in $N(y,k)h_\ell(z)$:

$$r_1(y) = y^k \ell^{k-1} \exp(-2\pi\ell y)$$
$$r_2(y) = y^k (\ell+1)^{k-1} \exp(-2\pi(\ell+1)y).$$

Note that, when $x = x_0 \pm \frac{1}{2}$, we have that

$$N(y,k)\,|h_\ell(z)| = \frac{(2\pi)^k}{\Gamma(k)}\left(r_1(y) + r_2(y)\right).$$

We additionally see that $r_1(y) + r_2(y) \geq \max(r_1(y), r_2(y))$. By elementary calculus, we note that r_1 is strictly increasing on $y_{\ell+1} \leq y \leq y_\ell$. Furthermore, r_2 is strictly decreasing for the same range of y values. Thus, $\max(r_1(y), r_2(y))$ is minimized at y_0, and $r_1 = r_2$ at y_0. Hence, if $x = x_0 \pm \frac{1}{2}$, recall (15) and Lemma 4.3 to gain the following:

$$N(y,k)\,|h_\ell(z)| \gg \frac{(2\pi y_0)^k}{\Gamma(k)}(\ell+1)^{k-1}\exp(-2\pi(\ell+1)y_0)$$

$$= \frac{(k-1)^k(\ell+1)^k}{\Gamma(k)(\ell+1)}\left|\log\left(1 - \frac{1}{\ell+1}\right)\right|^k \exp(-(k-1)) \cdot$$

$$\qquad \exp\left((k-1)(\ell+1)\left(\log\left(1 - \frac{1}{\ell+1}\right) + \frac{1}{\ell+1}\right)\right)$$

$$= \frac{e}{\ell+1}\left(\frac{k-1}{e}\right)^k \frac{1}{\Gamma(k)}\exp\left(-k \cdot M\left(\frac{1}{\ell+1}\right)\right).$$

Using (23), we derive (20) in this case.

■

It will be helpful to note that a minor modification of the proof of Lemma 4.4 gives

$$N(y,k)\,|h_\ell(z)| \ll \frac{\sqrt{k}}{\ell} \tag{26}$$

in V_ℓ as well.

Proof of Lemma 4.5. To begin, we note that $|\beta(z)| \leq |f_{\ell+2}(z)| + |f_{\ell-1}(z)| + |\varepsilon_1(z)| + |\varepsilon_2(z)| + |\delta(z)|$. Now, we break the proof into three parts:

Part 1: Consider $f_{\ell+2}(z)$ and $f_{\ell-1}(z)$. In the region V_ℓ, $N(y,k)\,|f_{\ell+2}(z)|$ has the greatest magnitude when $y = y_{\ell+1}$. Because of this, we use the substitution $y_{\ell+1} = \frac{k-1}{2\pi(\ell+1)}$ and Lemma 4.3 to obtain:

$$N(y,k)|f_{\ell+2}(z)| \leq \frac{(k-1)^k}{\Gamma(k)(\ell+2)}\exp(-(k-1))\left(1+\frac{1}{\ell+1}\right)^k\exp\left(-\frac{k-1}{\ell+1}\right)$$

$$= \frac{e}{\ell+2}\left(\frac{k-1}{e}\right)^k\frac{1}{\Gamma(k)}\exp\left(-k\cdot R\left(\frac{1}{\ell+1}\right)\right).$$

Then, using (23), we find that

$$N(y,k)|f_{\ell+2}(z)| \ll \frac{\sqrt{k}}{\ell}\exp\left(-k\cdot R\left(\frac{1}{\ell+1}\right)\right). \tag{27}$$

For $N(y,k)\,|f_{\ell-1}(z)|$, the function has the greatest magnitude when $y = y_\ell$ in the region V_ℓ. Then, we proceed in the same fashion as before to achieve the bound in (27) for $N(y,k)\,|f_{\ell-1}(z)|$. By similar methods, we find that $N(y,k)|f_{\ell+3}(z)| \leq N(y,k)|f_{\ell+2}(z)|$ and $N(y,k)|f_{\ell-2}(z)| \leq N(y,k)|f_{\ell-1}(z)|$.

Part 2: Let $\varepsilon_1(z)$ and $\varepsilon_2(z)$ be defined as in Section 4.2. By [RVY, p.18],

$$N(y,k)\,|\varepsilon_2(z)| \ll Q(k, 2\pi(\ell+3)y),$$

and

$$N(y,k)\,|\varepsilon_1(z)| \ll P(k, 2\pi(\ell-2)y),$$

where $Q(s,x)$ is the normalized incomplete gamma function and $P(s,x)$ is the complementary incomplete gamma function, defined by:

$$Q(s,x) = \frac{1}{\Gamma(s)}\int_x^\infty t^s e^{-t}\frac{dt}{t}; \qquad P(s,x) = \frac{1}{\Gamma(s)}\int_0^x t^s e^{-t}\frac{dt}{t}.$$

Using the results of [T], [RVY] derived

$$Q(s,x) \ll \exp\left(-\frac{(x-s)^2}{4s}\right) + \exp\left(-\frac{|x-s|}{4}\right). \tag{28}$$

Note that $P(s, x)$ is bounded above by (28) as well. Letting $s = k$ and $x = 2\pi(\ell + 3)y$ and using the inequality $\frac{k-1}{\ell+1} \leq 2\pi y \leq \frac{k-1}{\ell}$, we have

$$x - s = 2\pi(\ell + 3)y - k \geq \frac{2k}{\ell + 1} + O(1).$$

Thus, with (28), we derive

$$Q(k, 2\pi(\ell + 3)y) \ll \exp\left(-\frac{k}{(\ell + 1)^2}\right).$$

The same bound holds for $P(k, 2\pi(\ell - 2)y)$ as well. Since $R(t) < t^2$, the bound on $N(y, k)|\varepsilon_1(z)|$ and $N(y, k)|\varepsilon_2(z)|$ is consistent with (21).

Part 3: Recall $\delta(z)$ satisfies (11). Note that

$$\sum_{\substack{b|n \\ b<n}} \left(\frac{b}{n}\right)^{k-1} \leq \sum_{d>1} \frac{1}{d^{k-1}} = \zeta(k-1) - 1 \ll \frac{1}{2^k}. \tag{29}$$

The method of proof given in (26), Part 1, and Part 2, gives us the following bound in V_ℓ:

$$N(y, k) \sum_{n=1}^{\infty} |f_n(z)| \ll \frac{\sqrt{k}}{\ell}. \tag{30}$$

(Note that [RVY] utilizes the triangle inequality.) Thus, from (30) and (29), we find that

$$N(y, k)\, |\delta(z)| \ll \frac{\sqrt{k}}{2^k \ell}. \qquad \blacksquare$$

Proof of Lemma 4.3. Let $0 < t \leq \frac{1}{2}$. By the derivative of $R(t)$, we find that $R(t)$ is strictly increasing on its domain. Using its power series expansion, $R(t) \leq \frac{1}{2}t^2 < t^2$. To show that $M(t) < R(t)$, we define $w(t)$ to be the infinite sum $w(t) = \frac{1}{2}t + \frac{1}{3}t^2 + \frac{1}{4}t^3 + \cdots$. Then, $M(t) = w(t) - \log(1 + w(t))$, and our proof will be finalized as long as $w(t) < t$. Note that, for $t < \frac{5}{3}$,

$$w(t) \leq \frac{1}{2}t + \frac{1}{3}\left(\frac{t^2}{1-t}\right) < t. \qquad \blacksquare$$

5. Inequivalence of Zeros at Atkin-Lehner Cusps

In order to show inequivalence of zeros around certain cusps, we recall the theory of Atkin-Lehner involutions. We write $N = q_1 q_2 = QR$ such that $(Q, R) = 1$. An Atkin-Lehner involution W_Q^N is given by

$$W_Q^N = \begin{pmatrix} Qa & b \\ Nc & Qd \end{pmatrix}$$

where $a \equiv 1 \pmod{R}$, $b \equiv 1 \pmod{Q}$ and $\det W_Q^N = Q$, where $Q \neq 1$. Note additionally that $c \neq 0$, as $c = 0$ would require $Q = 1$, and would therefore imply $W_Q^N \in \Gamma_0(N)$.

Let χ_1' and χ_2' be primitive characters modulo q_1' and q_2', respectively. We define $q_1 = (q_1', R) \cdot (q_2', Q)$ and $q_2 = (q_1', Q) \cdot (q_2', R)$. Factor each character uniquely as $\chi_1' = \chi_1'^{(Q)} \chi_1'^{(R)}$ and $\chi_2' = \chi_2'^{(Q)} \chi_2'^{(R)}$, where $\chi_i'^{(Q)}$ has modulus (q_i', Q), and similarly for R. We define $\chi_1 = \chi_1'^{(R)} \chi_2'^{(Q)}$ and $\chi_2 = \chi_1'^{(Q)} \chi_2'^{(R)}$, and note that χ_i is a primitive character modulo q_i.

Weisinger showed in [W] that this implies

$$E_{\chi_1', \chi_2', k}(z)\big|_{W_Q^N} = c_Q E_{\chi_1, \chi_2, k}(z)$$

for some constant c_Q. This allows one to study the behavior of $E_{\chi_1, \chi_2, k}(z)$ around the cusp $\frac{a}{Rc}$ in terms of the series $E_{\chi_1', \chi_2', k}(z)$ around the cusp at infinity. We will use this theory in the coming proposition.

Let $\mathcal{K}(q_2)$ denote the region

$$\mathcal{K}(q_2) = \left\{ z = x + iy \in \mathbb{H} : y > \frac{1}{q_2}, \ -\frac{1}{2} < x \le \frac{1}{2} \right\}.$$

Proposition 5.1. *For an Atkin-Lehner involution W_Q^N, the image of $\mathcal{K}(q_2')$ under the involution W_Q^N is disjoint with the region $\mathcal{K}(q_2)$.*

Proof. Let $z = x + iy \in \mathcal{K}(q_2')$. Then

$$\mathrm{Im}\left(W_Q^N z \right) = \frac{Qy}{(Ncx + Qd)^2 + (Ncy)^2} \le \frac{Qy}{N^2 c^2 y^2}$$

$$= \frac{1}{Rc^2 Ny} < \frac{q_2'}{Rc^2 N} \le \frac{q_2'}{RN}.$$

Note that

$$\frac{q_2'}{RN} \le \frac{1}{q_2},$$

since

$$\frac{q_2' q_2}{RN} = \frac{q_2}{q_1' R} = \frac{(q_1', Q)}{q_1'} \cdot \frac{(q_2', R)}{R} \le 1.$$

Therefore $\mathrm{Im}(W_Q^N z) \notin \mathcal{K}(q_2)$. $\blacksquare$

Corollary 5.2. *The zeros of $E_{\chi_1, \chi_2, k}(z)$ in $\mathcal{K}(q_2)$ are $\Gamma_0(q_1 q_2)$-inequivalent to the image of the zeros of $E_{\chi_1', \chi_2', k}(z)$ under the Atkin-Lehner involution W_Q^N.*

Acknowledgements

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