An Efficient Way of Understanding the Maximum Number of Steady States of Chemical Reaction Networks

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Schedule

- Backgrounds
- Results
- Questions

Backgrounds

Main Question

When do mixed volume equal to the maximum number of positive steady states?

Why do we care?

- Calculating mixed volume is faster.
- There are CRN that hold the property.
- It hasn't been looked at by others that much.

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Chemical Reaction Network: Model of chemical reactions happening around us.

$$H_2 + O \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} H_2O \qquad \longrightarrow \qquad 2A+B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C$$

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Derived ODE:

$$\begin{aligned} a' &= -k_1 a^2 b + k_2 c \\ b' &= -k_1 a^2 b + k_2 c \\ c' &= k_1 a^2 b - k_2 c \end{aligned}$$

Conservation Law:

$$\begin{aligned} b - a &= T \\ a + c &= S \end{aligned}$$

Mixed volume: Number of nonzero complex solutions to polynomial system.

$$\mathrm{MV}(p_1, p_2, \ldots p_n) = \sum_{J \subseteq \{1,2,\ldots,n\}} (-1)^{n-\#J} \mathrm{Vol}(\sum_{j \in J} p_j)$$ [6]

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Fully Reversible Network: All the product complexes are also reactant complex and vice versa.

Irreversible Network: There is no way in the network to reach to starting reactant complexes from product complexes.

False Conjecture

Conjecture

If mixed volume equals to number of positive steady states in a irreversible CRN, then making it fully reversible does not change equality.

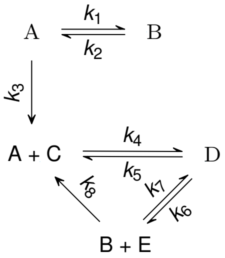

Figure: Mixed volume $\neq$ positive steady states

Solution$\approx$ $(8.575^{-11} - 1.136^{-10}i, 8.575^{-11} - 1.136^{-10}, -1.459^{8} + 1.480^{8}i, \mathbf{1}, -410.656 - 401.14i)$

Monomolecular CRN

$$nA \xrightarrow{k_1} (n-1)A \xrightarrow{k_2} \cdots \xrightarrow{k_{i-1}} mA \xrightarrow{k_i} (m-1)A$$

Figure: Generalized Monomolecular Irreversible Network

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The ODE we get from the CRN in Figure 1 is given by,

$$\frac{da}{dt} = -k_1 a^n - k_2 a^{n-1} - \cdots - k_i a^m$$

We can get the solutions for a by setting this ODE to zero,

$$-k_1 a^n - k_2 a^{n-1} - \cdots - k_i a^m = 0$$

$$-a^m(k_n a^{n-m} + k_{n-1} a^{n-m-1} + \cdots + k_i a^0) = 0$$

By Fundamental Theorem of Algebra

m degenerate solutions where $a = 0$ and $n - m$ nonzero complex solutions or mixed volume.

Monomolecular Networks

Descartes' Rule

The number of positive solutions to a univariate polynomial $\leq$ number of sign changes when it's expanded with respect to its variable.

Assume the derived univariate polynomial has alternating signs,

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Theorem

If the number of sign changes in the simplified polynomial acquired from the ODE equal to mixed volume, then the maximum number of steady states equal to mixed volume.

Example:

$$\mathrm{A} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 0 \xrightarrow{k_3} 2\mathrm{A} \xrightarrow{k_4} 3\mathrm{A}$$

ODE:

$$\frac{da}{dt} = (k_1 + 2k_3) - k_2 a + k_4 a^2$$

Solve:

$$(k_1 + 2k_3) - k_2 a + k_4 a^2 = 0$$

Substitute $k_1 = 1, k_2 = 3, k_3 = 0.5, k_4 = 1$:

$$a = \{2, 1\}$$

Bimolecular Networks

$$\mathrm{A} + 2\mathrm{B} \xrightarrow{k_1} 2\mathrm{A} + 3\mathrm{B}$$
$$2\mathrm{A} + 2\mathrm{B} \xrightarrow{k_2} \mathrm{A} + \mathrm{B}$$

Figure: Bimolecular chemical reaction network

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ODE's and conservation law:

$a' = k_1 ab^2 - k_2 a^2 b^2$

$b' = k_1 ab^2 - k_2 a^2 b^2$

$a - b = T \qquad \text{for } T \in \mathbb{R}^+$

$a = T + b$

By substitution,

$$k_1(T + b)b^2 - k_2(T + b)^2 b^2 = 0$$

Assuming $k_1 = 1, k_2 = 0.5, T = 1$ we get,

$$(a, b) = \{(-1, 0), (1, 2), (0, 1), (0, 1)\}$$
$$\therefore \text{Mixed volume} = 1, \text{Steady state} = 1$$

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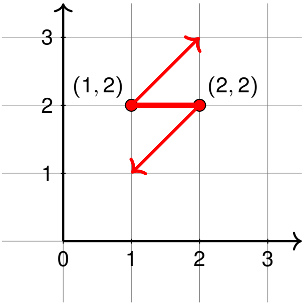

Figure: Reaction graph: Flipped version of Harry Potter's scar?

Reactant Polytope: Convex hull polytope created only by the reactants' exponent vector.

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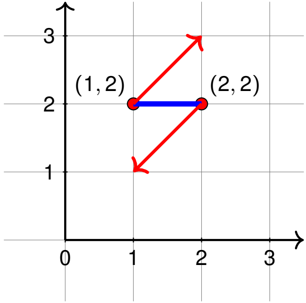

Figure: Reactant Polytope: Flipped version of Harry Potter's scar?

Further Research

- Can we prove that regardless of the number of species in a chemical reaction network, if only one reactant species changes in the reactant complex while the other species are constant, then the maximum number of positive steady states would be equal to number of sign changes?

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- Is there at all any fully reversible network that hold the property?

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- Can we prove that regardless of the number of species in a chemical reaction network, if only one reactant species changes in the reactant complex while the other species are constant, then the maximum number of positive steady states would be equal to number of sign changes?
- Is there at all any fully reversible network that hold the property?
- Can there be network whose reactant species are all inconsistant throughout the network but mixed volume equals to maximum number of positive steady states?

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